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A Unified Framework for the Algebra of Unsharp Quantum Mechanics

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On the basis of the concrete operations definable on the set of effect operators on a Hilbert space, an abstract algebraic structure of sum Brouwer–Zadeh (SBZ)-algebra is introduced. This structure consists of a partial sum operation and two mappings which turn out to be Kleene and Brouwer unusual orthocomplementations. The Foulis–Bennett effect algebra substructure induced by any SBZ-algebra, allows one to introduce the notions of unsharp "state" and "observable" in such a way that any "state–observable" composition is a standard probability measure (classical state). The Cattaneo–Nisticò BZ substructure induced by any SBZ-algebra permits one to distinguish, in an equational and simple way, the sharp elements from the really unsharp ones. The family of all sharp elements turns out to be a Foulis–Randall orthoalgebra. Any unsharp element can be "roughly" approximated by a pair of sharp elements representing the best sharp approximation from the bottom and from the top respectively, according to an abstract generalization introduced by Cattaneo of Pawlack "rough set" theory (a generalization of set theory, complementary to fuzzy set theory, which describes approximate knowledge with applications in computer sciences). In both the concrete examples of fuzzy sets and effect operators the "algebra" of rough elements shows a weak SBZ structure (weak effect algebra plus BZ standard poset) whose investigation is set as an interesting open problem.

1. THE "ALGEBRA" OF UNSHARP QUANTUM MECHANICS ON HILBERT SPACES: METATHEORETICAL PRINCIPLES

The "logic" of *quantum simple propositions* in conventional quantum mechanics (QM) based on a complex Hilbert space $\mathcal{H}$ is realized by the set $\mathcal{M}(\mathcal{H})$ of all subspaces (i.e., closed linear manifolds) of $\mathcal{H}$. The set $\mathcal{M}(\mathcal{H})$, equipped with the set-theoretic inclusion $\subseteq$ and the orthocomplementation $^\perp$ assigning to any subspace M its annihilator $M^\perp$, is an orthocomplemented

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orthomodular atomic complete lattice (the so-called "quantum logic"), bounded by the trivial subspaces $\{\underline{0}\}$ and $\mathcal{H}$; the g.l.b. of any family $\{M_j\}$ of subspaces is the set-theoretic intersection (i.e., $\wedge M_j = \cap M_j$), the l.u.b. of the same family is the subspace generated by the set-theoretic union (i.e., $\vee M_j = (\cup M_j)^{\perp\perp}$). All this can be summarized by the structure

$$\langle \mathcal{M}(\mathcal{H}), \wedge, \vee, {}^{\perp}, \{\underline{0}\}, \mathcal{H}\rangle \tag{1.1a}$$

The collection of all *quantum events* is realized by the set $\Pi(\mathcal{H})$ of all orthogonal projections on $\mathcal{H}$, which has a structure

$$\langle \Pi(\mathcal{H}), \leq, {}', \mathrm{O}, \mathrm{I}\rangle \tag{1.1b}$$

of an orthomodular orthocomplemented atomic complete lattice with respect to the *phenomenological* partial ordering defined for P_1, $P_2 \in \Pi(\mathcal{H})$ by

$$P_1 \leq P_2 \quad \text{iff} \quad \forall \varphi \in \mathcal{H}, \qquad \langle \varphi | P_1 \varphi\rangle \leq \langle \varphi | P_2 \varphi\rangle \tag{or}$$

and the orthocomplementation on $\Pi(\mathcal{H})$ defined according to one of the equivalent forms

$$P' := 1 - P = P_{\mathrm{Ran}(P)^{\perp}} = P_{\mathrm{Ker}(P)} \tag{oc}$$

[denoting by P_M the orthogonal projection which projects onto the subspace $M \in \mathcal{M}(\mathcal{H})$]. Vectors from $\mathcal{H}_0 := \mathcal{H} \setminus \{\underline{0}\}$ are interpreted as (pure) *preparation procedures* of individual samples of the physical entity under well defined and repeatable conditions. For any preparation $\varphi \in \mathcal{H}_0$ and any event $P \in \Pi(\mathcal{H})$ the quantity

$$\mu(\varphi, P) := \frac{\langle \varphi | P \varphi\rangle}{\|\varphi\|^2} \in [0, 1] \tag{1.2}$$

is the *probability of the occurrence* of the answer "yes" for the event P when the entity is prepared in φ.

The one-to-one mapping $\mathcal{M}(\mathcal{H}) \mapsto \Pi(\mathcal{H})$, $M \to P_M$, associating with the subspace M of $\mathcal{H}$ the orthogonal projection P_M which projects onto M, is an isomorphism in the category of orthomodular lattices. In axiomatic quantum mechanics, it is assumed that P_M represents the *event* which measures the quantum simple proposition M. The *certainly-yes domain* of any event $P \in \Pi(\mathcal{H})$ is defined as $D_1(P) := \{\psi \in \mathcal{H}_0 \colon \mu(\psi, P) = 1\}$, i.e., the collection of all preparations in which the answer "yes" to the event P occurs with certainty (probability one); trivially, the subspace onto which P projects is $M_1(P) := \mathrm{Ker}\,(1 - P) = D_1(P) \cup \{\underline{0}\}$.

Summarizing, the now outlined mathematical realization of sharp QM based on a Hilbert space is founded on the identification between "events" (mathematically realized by orthogonal projections) and "simple propositions" (mathematically realized by subspaces):

$$\begin{array}{ccc} \boxed{\text{EVENT}} & & \boxed{\text{SIMPLE PROPOSITION}} \\ P & \leftrightarrow & M_1(P) \end{array}$$

Unsharp QM is an enlargement of the above theory, in which effect operators (i.e., linear operators F on $\mathcal{H}$ such that $\forall\varphi \in \mathcal{H}$, $0 \leq \langle\varphi|F\varphi\rangle \leq \|\varphi\|^2$) represent the extension of orthogonal projections and orthopairs of subspaces [i.e., pairs of subspaces (M_1, M_0) of $\mathcal{H}$ which are mutually orthogonal: $M_1 \perp M_0$] generalize the standard notion of subspace.

Let us denote by $\mathscr{E}(\mathcal{H})$ the set of all effect operators; then $\mathscr{E}(\mathcal{H})$ strictly contains the set $\Pi(\mathcal{H})$ of all orthogonal projections. As usual, vectors from $\mathcal{H}_0 := \mathcal{H} \setminus \{\underline{0}\}$ are interpreted as *preparation procedures*, and for any effect $F \in \mathscr{E}(\mathcal{H})$ and any preparation $\varphi \in \mathcal{H}_0$ the quantity

$$\mu(\varphi, F) := \frac{\langle\varphi|F\varphi\rangle}{\|\varphi\|^2} \in [0, 1] \tag{1.3}$$

is the *probability of the occurrence* of the answer "yes" for the effect F when the entity is prepared according to φ.

The set of all Hilbert space unsharp *quantum propositions* is the family $L_f(\mathcal{H},\perp) := \{(M_1, M_0): M_1, M_0 \in \mathcal{M}(\mathcal{H}), M_1 \perp M_0\}$ of all orthopairs of simple propositions (=subspaces) of $\mathcal{H}$. For any effect F we introduce the two mutually orthogonal subspaces of $\mathcal{H}$: $M_1(F) = \mathrm{Ker}(1 - F)$ [identified with the *certainly-yes domain of* F, $D_1(F) := \{\psi \in \mathcal{H}: \mu(\psi, F) = 1\} = M_1(F)\setminus\{\underline{0}\})$] and $M_0(F) = \mathrm{Ker}(F)$ [identified with the *certainly-no domain of* F, $D_0(F) := \{\varphi \in \mathcal{H}: \mu(\varphi, F) = 0\} = M_0(F)\setminus\{\underline{0}\}$]. Vectors of the certainly-yes (resp., no) domain $D_1(F)$ [resp., $D_0(F)$], represent preparations with respect to which the answer "yes" (resp., "no") to the effect F (resp., $F' := 1 - F$) is certain, i.e., the probability of the occurrence of F (resp., F') is 1. Therefore, we have the following unsharp extension of the above diagram:

$$\begin{array}{ccc} \boxed{\text{EFFECT}} & & \boxed{\text{PROPOSITION}} \\ F & \rightarrow & (M_1(F), M_0(F)) \end{array}$$

in which the identification between the orthomodular lattices of projectors and subspaces is broken up into a mapping from the family of effects onto the family of propositions, which is not one-to-one.

Remark 1.1. In the exact case of an orthogonal projection $P \in \Pi(\mathcal{H})$, the proposition $(M_1(P), M_0(P))$ is equal to the proposition $(M_1(P), M_1(P)^\perp)$, which can be identified with the simple proposition $M_1(P)$.

Several relations and operations can be introduced on the set of effect operators on a Hilbert space. We list those which can be considered as relevant:

(1) *The binary relation of partial ordering*, which is the natural extension of the (*or*): for $F_1, F_2 \in \mathscr{E}(\mathscr{H})$

$$F_1 \leq F_2 \quad \text{iff} \quad \forall \varphi \in \mathscr{H}, \qquad \langle \varphi | F_1 \varphi \rangle \leq \langle \varphi | F_2 \varphi \rangle \tag{$\mathscr{E}$-or}$$

The structure $\langle \mathscr{E}(\mathscr{H}), \leq, O, 1 \rangle$ is a poset which is not a lattice [for an indirect prove of this statement see Davies (1976) and for a *direct* one see Greechie and Gudder (n.d.)].

(2) *The binary relation of orthogonality on effects is*

$$F_1 \perp F_2 \quad \text{iff} \quad F_1 + F_2 \leq 1$$

Once we define the set $(\mathscr{E}(\mathscr{H}) \times \mathscr{E}(\mathscr{H}))_\perp := \{(F_1, F_2) \in \mathscr{E}(\mathscr{H}) \times \mathscr{E}(\mathscr{H}): F_1 \perp F_2\}$, we can introduce:

(3) *The binary operation of partial sum on* effects $\oplus$: $(\mathscr{E}(\mathscr{H}) \times \mathscr{E}(\mathscr{H}))_\perp \mapsto \mathscr{E}(\mathscr{H})$, defined as

$$F_1 \oplus F_2 := F_1 + F_2 \quad \text{iff} \quad (F_1, F_2) \in (\mathscr{E}(\mathscr{H}) \times \mathscr{E}(\mathscr{H}))_\perp$$

(Note that this definition makes use of the previous structure of poset with orthogonality $\langle \mathscr{E}(\mathscr{H}), \leq, \perp O, 1 \rangle$).

(4) *The unary operation of Kleene orthocomplementation*, which is the natural extension of the first equality in (*oc*): $\forall F \in \mathscr{E}(\mathscr{H})$

$$F' := 1 - F \tag{K-oc}$$

(5) *The unary operation of Brouwer orthocomplementation*, which is the extension of the second equality in (*oc*): $\forall F \in \mathscr{E}(\mathscr{H})$

$$F^{\sim} := P_{\mathrm{Ker}(F)} \tag{B-oc}$$

The above are the relations and operations which we shall consider in the sequel. They are not the unique one that can be introduced on $\mathscr{E}(\mathscr{H})$; for instance, we can also quote:

(6) *The "convex" product*, which can be stated as the external operation

$$\cdot : \quad [0, 1] \times \mathscr{E}(\mathscr{H}) \mapsto \mathscr{E}(\mathscr{H}), \qquad (\lambda, F) \to \lambda \cdot F$$

associating with any number $0 \leq \lambda \leq 1$ and any effect F the new effect $\lambda \cdot F \in \mathscr{E}(\mathscr{H})$.

Let us note that owing to (2) and (6), $\mathscr{E}(\mathscr{H})$ turns out to be a convex set, with respect to the following operation:

(2-6) *The convex "combination"*

$$c: \quad [0, 1] \times \mathscr{E}(\mathscr{H}) \times \mathscr{E}(\mathscr{H}) \mapsto \mathscr{E}(\mathscr{H}),$$

$$(\lambda, F, G) \to c(\lambda, F, G) := \lambda F + (1 - \lambda) G$$

[operation which can be extended to σ-convex combination in a trivial way].

(7) *The "filtering" product*, which can be stated as the internal operation

$$\odot: \quad \mathscr{E}(\mathscr{H}) \times \mathscr{E}(\mathscr{H}) \mapsto \mathscr{E}(\mathscr{H}), \quad (F, G) \to F \odot G := G^{1/2} \circ F \circ G^{1/2}$$

Adopting the notation of Davies (1976), for any effect $F \in \mathscr{E}(\mathscr{H})$, the map T_F: $\mathscr{T}(\mathscr{H}) \mapsto \mathscr{T}(\mathscr{H})$ from the set of all trace-class operators into itself defined, $\forall \rho \in \mathscr{T}(\mathscr{H})$, by $T_F(\rho) := F^{1/2} \circ \rho \circ F^{1/2}$ is a pure operation (linear, positive, absorbing, and pure transformation) describing a physical "filter." For every pure state (induced by the preparation $\psi \in \mathscr{H}_0$) $\rho_\psi :=, |\psi\rangle\langle\psi|/\|\psi\|^2$ we have that $\mathrm{tr}[T_F(\rho_\psi)] = \langle\psi|F\psi\rangle/\|\psi\|^2 = \mu(\psi, F)$, i.e., F is the effect realized by the pure operation T_F ["The effect F determines the probability of transmission, but not the form of the transmitted state" (Davies, 1979)].

Of course, the composition of two pure operations $T_F \circ T_G$ acts on trace-class operators in the following way: $(T_F \circ T_G)\,(\rho) = F^{1/2} \circ G^{1/2} \circ \rho \circ G^{1/2} \circ F^{1/2}$ and then

$$\begin{aligned} \mathrm{tr}[(T_F \circ T_G)(\rho_\psi)] &= \mathrm{tr}[G^{1/2} \circ F \circ G^{1/2} \circ \rho_\psi] \\ &= \frac{\langle\psi|(G^{1/2} \circ F \circ G^{1/2})\psi\rangle}{\|\psi\|^2} = \mu(\psi, G^{1/2} \circ F \circ G^{1/2}) \end{aligned}$$

i.e., $F \odot G$ is the effect which realizes the pure operation $T_F \circ T_G$.

In conclusion, (6) allows one to introduce the convex structure of effect algebras [which is not considered in the structures based on the (1)–(5)] and (7) expresses a realization by effects of the composition of particular pure filtering operations. In some sense, in any algebraic approach based on (1)–(5) [without considering (6) and (7)] something of the "physics" of effects is definitively lost.

A number of algebraic structures have recently been proposed as an adequate abstraction of the effects of a Hilbert space [BZ-posets (Cattaneo and Nisticò, 1989), D-posets (Kopka and Chovanec, 1994), effect algebras (Foulis and Bennett, 1994), quantum MV algebras (Giuntini, 1995), where each of them mimics, as an abstract axiomatization, only a part of the above operations (1)–(7). The problem of the "adequate" algebraic structure to describe unsharpness in quantum theory can be discussed on the basis of some *metatheoretical principles*, stated outside the mathematics of the involved structure. We assume the following as a suitable *minimal* choice for a further discussion and, if the case, criticism:

(MT_1) The abstract algebraic structure $\mathscr{E}$ describing unsharpness in QM must have as concrete mathematical model the family $\mathscr{E}(\mathscr{H})$ of effect operators of usual unsharp (generalized) QM based on a Hilbert space $\mathscr{H}$.

(MT_2) Making use of the formal structure of $\mathscr{E}$, it must be possible to introduce a class of mathematical objects interpreted as "observables" and a class of mathematical objects interpreted as "states". In the Hilbert space model observables must correspond to usual POV measures and states to density operators (at least for Hilbert space whose dimension is greater than two).

(MT_3) In the mathematical structure of $\mathscr{E}$, which describes a general situation of unsharpness, a subclass $\mathscr{E}_s$ of elements, interpreted as "sharp" ("exact," "crisp"), and distinguished from elements in $\mathscr{E}\backslash\mathscr{E}_s$, interpreted as strictly "unsharp" ("fuzzy"), must be singled out (preferably in an equational and simple way).

In the Hilbert space model of effect operators this class must correspond to the family of orthogonal projections.

(MT_4) For any element $a \in \mathscr{E}$ there must exist two sharp ones a_*, $a^* \in \mathscr{E}_s$ which represent the "best" sharp approximation from the bottom and the "best" sharp approximation from the top of a. The pair (a_*, a^*) is the "rough" approximation of a, which for a sharp element $a \in \mathscr{E}_s$ must coincide with the pair (a, a) [see Cattaneo (1996) for an introduction to rough set theory, and for its algebraic generalization].

A little comment about (MT_3) [and its "consequence" (MT_4)]: Unsharpness has some meaning only if compared with sharpness (and this is possible only if the structure permits one to distinguish sharp from unsharp elements); there is no unsharpness without sharpness.

2. SUM BROUWER–ZADEH ALGEBRAS AS A PROPOSAL FOR EFFECT ALGEBRAS AND UNSHARP QUANTUM LOGICS

The above metatheoretical principles suggest to consider as a sufficiently fruitful minimal algebraic abstraction of effect operators a structure of *sum Brouwer–Zadeh (SBZ)-algebra*

$$\langle \mathscr{E}, \perp, \oplus, '; {}^{\sim}, 0, 1 \rangle$$

where:

(i) $\perp$ is a binary relation on the set $\mathscr{E}$ containing at least two distinguished elements 0 and 1 ($0 \neq 1$) [in the sequel we denote by $a \perp b$ the fact that the pair $(a, b) \in \mathscr{E} \times \mathscr{E}$ belongs to the binary relation $\perp$ and by $(\mathscr{E} \times \mathscr{E})_\perp \subseteq \mathscr{E} \times \mathscr{E}$ the collection of all such pairs] such that:

(og-1) [*Symmetry law*]

$$a \perp b \quad \text{implies} \quad b \perp a$$

(og-2) [*Regularity law*]

$$a \perp a \quad \text{and} \quad b \perp b \quad \text{imply} \quad a \perp b$$

(og-3) [Zero-one law]

$$1 \perp a \quad \text{implies} \quad a = 0$$

(ii) $\oplus$: $(\mathscr{E} \times \mathscr{E})_\perp \mapsto \mathscr{E}$ is a partial operation defined on pairs of mutually orthogonal elements from $\mathscr{E}$ such that the following hold:

(sa-1) [*Commutative law*] If $a \perp b$, then $b \perp a$ is a consequence of the symmetry property of $\perp$, and

$$a \oplus b = b \oplus a$$

(sa-2) [*Associative law*] If $a \perp b$ and $(a \oplus b) \perp c$, then $b \perp c$, $a \perp (b \oplus c)$, and

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c$$

(iii) $'$: $\mathscr{E} \mapsto \mathscr{E}$ is a unary operation on $\mathscr{E}$ such that the following hold:
(koc-1) [*K-Orthosupplementation law*]

$$a \perp a' \quad \text{and} \quad a \oplus a' = 1$$

(koc-2) [*K-Uniqueness law*]

$$a \perp b \quad \text{and} \quad a \oplus b = 1 \quad \text{imply} \quad b = a'$$

(iv) $^\sim$: $\mathscr{E} \mapsto \mathscr{E}$ is a unary operation on $\mathscr{E}$ such that the following hold:
(boc-1) [*B-Symmetry law*]

$$\exists r : a \oplus r = b^\sim \quad \text{implies} \quad \exists s : b \oplus s = a^\sim$$

(boc-2) [*B-Orthogonality law*]

$$a \perp a^\sim$$

(boc-3) [*B-Noncontradiction law*]

$$\exists r : a^\sim \oplus r = c \quad \text{and} \quad \exists s : a^{\sim\sim} \oplus s = c \quad \text{imply} \quad c = 1$$

Example 2.1. The real unit interval is an SBZ-algebra $\langle [0, 1], \perp, \oplus, ', {}^\sim, 0, 1 \rangle$ with respect to (1) the orthogonality relation: let r, s, $\in [0, 1]$; then $r \perp s$ iff $r + s \leq 1$; (2) the partial sum operation: let $r, s \in [0, 1]$ be such that $r \perp s$, then $r \oplus s := r + s$ ($+$ denotes the standard sum operation of real numbers); (3) the K-orthocomplementation: $\forall r \in [0, 1]$, $r' := 1 - r$; (4) the B-orthocomplementation: $\forall r \in [0, 1]$, $r^\sim = 1$ if $r^\sim = 0$ and $r^\sim = 0$ if $r \neq 0$.

Example 2.2. The unsharp (fuzzy) set theory on the reference space U is an SBZ-algebra $\langle [0, 1]^U, \perp, \oplus, ', {}^\sim, \underline{0}, \underline{1} \rangle$ with respect to: (1) the orthogonality relation: let f, $g \in [0, 1]^U$; then $f \perp g$ iff $f + g \leq \underline{1}$ (where $\underline{1}$ associates with any $x \in U$ the number $\underline{1}(x) := 1 \in [0, 1]$); (2) the partial sum operation: let

$f, g \in [0, 1]^U$ be such that $f \perp g$; then $f \oplus g := f + g$; (3) the K-orthocomplementation: $\forall f \in [0, 1]^U$, then $f' = \underline{1} - f$; (4) the B-orthocomplementation: $\forall f \in [0, 1]^U$, then $f^{\sim} = \chi_{A_0(f)}$ [where $A_0(f) := \{x \in U: f(x) = 0\}$, and for any subset $A \subseteq U$, $\chi_A(x) = 1$ if $x \in A$ and $= 0$ otherwise].

Example 2.3. The "classical logic" of a measurable space $(\mathbb{K}, \mathcal{B}(\mathbb{K}))$. Let $\mathcal{B}(\mathbb{K}))$ be a σ-algebra on the nonempty set $\mathbb{K}$; the structure $\langle \mathcal{B}(\mathbb{K}), \perp, \oplus, ', ^{\sim}, \emptyset, 1\rangle$ is an SBZ-algebra with respect to: (1) the orthogonality relation: let $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{K})$; then $\Delta_1 \perp \Delta_2$ iff $\Delta_1 \cap \Delta_2 = \emptyset$; (2) the partial sum operation: let $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{K})$ be such that $\Delta_1 \perp \Delta_2$; then $\Delta_1 \oplus \Delta_2 := \Delta_1 \cup \Delta_2$ (in this case we also write $\Delta_1 \uplus \Delta_2$); (3) the K-orthocomplementation: $\forall \Delta \in \mathcal{B}(\mathbb{K})$, $\Delta' := \mathbb{K}\setminus\Delta$; (4) the B-orthocomplementation: $\forall \Delta \in \mathcal{B}(\mathbb{K})$, $\Delta^{\sim} := \mathbb{K}\setminus\Delta = \Delta'$.

Example 2.4. The unsharp quantum mechanics on the Hilbert space $\mathcal{H}$ is an SBZ-algebra $\langle \mathcal{E}(\mathcal{H}), \perp, \oplus, ', ^{\sim}, O, 1\rangle$ with respect to: (1) the orthogonality relation: let $F, G \in \mathcal{E}(\mathcal{H})$; then $F \perp G$ iff $F + G \leq 1$; (2) the partial sum operation: let $F, G \in \mathcal{E}(\mathcal{H})$ be such that $F \perp G$; then $F \oplus G := F + G$; (3) the K-orthocomplementation: $\forall F \in \mathcal{E}(\mathcal{H})$, $F' := 1 - F$; (4) the B-orthocomplementation: $\forall F \in \mathcal{E}(\mathcal{H})$, $F^{\sim} := P_{\mathrm{Ker}(F)}$.

Example 2.5. The standard "quantum logic" of a Hilbert space $\mathcal{H}$ is an SBZ-algebra $\langle \mathcal{M}(\mathcal{H}), \perp, \oplus, ', ^{\sim}, \{\underline{0}\}, \mathcal{H}\rangle$ with respect to: (1) the orthogonality relation: let $M_1, M_2 \in \mathcal{M}(\mathcal{H})$; then $M_1 \perp M_2$ iff $\forall \psi_1 \in M_1$ and $\forall \psi_2 \in M_2$, $\langle \psi_1 | \psi_2 \rangle = 0$; (2) the partial sum operation: let $M_1, M_2 \in \mathcal{M}(\mathcal{H})$ be such that $M_1 \perp M_2$; then $M_1 \oplus M_2 := M_1 \vee M_2$ (i.e., the subspace generated by $M_1 \cup M_2$); (3) the K-orthocomplementation: $\forall M \in \mathcal{M}(\mathcal{H})$, $M' := M^{\perp}$; (4) the B-orthocomplementation: $\forall M \in \mathcal{M}(\mathcal{H})$, $M^{\sim} := M^{\perp} = M'$.

As a consequence of these examples we can state the following result.

Conclusion 1. The structure of SBZ-algebra satisfies the metatheoretical principle (MT_1) since the unsharp QM of effect operators on a Hilbert space is an SBZ-algebra (Example 2.4).

Definition 2.1. Let $\mathcal{E}$ and $\mathcal{F}$ be two SBZ-algebras. A mapping $\phi: \mathcal{E} \mapsto \mathcal{F}$ is a *morphism* iff the following hold:

(i) $\phi(1_{\mathcal{E}}) = 1_{\mathcal{F}}$;
(ii) let $a, b \in \mathcal{E}$ be such that $a \perp b$; then $\phi(a) \perp \phi(b)$;
(iii) let $a, b \in \mathcal{E}$ be such that $a \perp b$; then $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$;
(iv) $\phi(a^{\sim}) = \phi(a)^{\sim}$.

A morphism is called *epimorfism* iff onto.

Proposition 2.1. If $\mathcal{E}$ and $\mathcal{F}$ are SBZ-algebras and $\phi: \mathcal{E} \mapsto \mathcal{F}$ is a morphism, then:

(i-a) $\phi(0_{\mathscr{E}}) = 0_{\mathscr{F}}$.

(iii-a) Let $a, b \in \mathscr{E}$; if $\exists s \in \mathscr{E}$ s.t. $a \perp s$ and $a \oplus s = b$, then the element $\phi(s) \in \mathscr{F}$ is s.t. $\phi(s) \perp \phi(a)$ and $\phi(a) \oplus \phi(s) = \phi(b)$.

(iii-b) $\phi(a') = \phi(a)'$.

Definition 2.2. A mapping $\phi: \mathscr{E} \mapsto \mathscr{F}$ from the SBZ-algebra $\mathscr{E}$ into the SBZ-algebra $\mathscr{F}$ is a *po-morphism* iff the above properties (i), (ii) of Definition 2.1, plus the following conditions hold:

(iii-a) Let $a, b \in \mathscr{E}$; then $\exists s \in \mathscr{E}$: $a \perp s$ and $a \oplus s = b$ imply $\exists r \in \mathscr{F}$: $r \perp \phi(a)$ and $\phi(a) \oplus r = \phi(b)$ [in general $r \neq \phi(s)$].

(iii-b) $\phi(a') = \phi(a)'$.

(iv) $\phi(a^{\sim}) = \phi(a)^{\sim}$.

A *po-epimorphism* is a po-morphism which is onto.

Any morphism is trivially a po-morphism (see Proposition 2.1).

Definition 2.3. Let $\mathscr{E}$ and $\mathscr{F}$ be two SBZ-algebras. A mapping $\phi: \mathscr{E} \mapsto \mathscr{F}$ is an *isomorphism* iff it is an epimorphism which satisfies further:

(v) Let $a, b \in \mathscr{E}$; then $\exists r \in \mathscr{E}$: $\phi(a) \perp r$ and $\phi(a) \oplus r = \phi(b)$ implies $\exists s \in \mathscr{E}$: $a \perp s$ and $a \oplus s = b$.

Proposition 2.2. If $\phi: \mathscr{E} \mapsto \mathscr{F}$ is an isomorphism, then ϕ is a bijection and $\phi^{-1}: \mathscr{F} \mapsto \mathscr{E}$ is an isomorphism.

2.1. The Foulis–Bennett Effect Algebra as a Substructure of SBZ-Algebra: States and Observables

Neglecting the unary mapping $\sim$ in the above definition of SBZ-algebra, points *(i)–(iii)* define in an equivalent way a structure of a regular "effect" algebra, according to Foulis and Bennett (1994).

Theorem 2.1. Let $\langle \mathscr{E}, \perp, \oplus, ', {}^{\sim}, 0, 1 \rangle$ be an SBZ-algebra. Then the substructure $\langle \mathscr{E}, \oplus, 0, 1 \rangle$, is a regular FB-effect algebra, i.e., a set $\mathscr{E}$ with two special elements 0, 1 and a binary operation $\oplus$ partially defined on $\mathscr{E}$ satisfying for all $a,b,c, \in \mathscr{E}$ the following conditions:

(sa-1) [*Commutative law*] If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(sa-2) [*Associative law*] If $a \oplus b$ is defined and $(a \oplus b) \oplus c$ is defined, then $b \oplus c$, $a \oplus (b \oplus c)$ are defined, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(sa-3) [*Orthosupplementation law*] For every $a \in \mathscr{E}$ there exists a unique $a' \in \mathscr{E}$ such that $a \oplus a'$ is defined and $a \oplus a' = 1$.

(sa-4) [*Zero–One law*] If $1 \oplus a$ is defined, then $a = 0$.

(re) [*Regularity law*] If $a \oplus a$ and $b \oplus b$ are defined, then $a \oplus b$ is defined, too.

Definition 2.4. Let $\mathscr{E}$ be a regular FB-effect algebra. Let $a, b \in \mathscr{E}$; then we define:

(i) The binary relation

$$a \perp b \quad \text{iff} \quad a \oplus b \text{ is defined} \qquad (og\text{-}\mathscr{E})$$

and in this case we say that a is *orthogonal* to b.

(ii) The binary relation

$$a \leq b \quad \text{iff} \quad \exists c \in \mathscr{E} : a \perp c \quad \text{and} \quad a \oplus c = b \qquad (or\text{-}\mathscr{E})$$

and in this case we say that *a is less than or equal to b.*

The proof of the following Theorem can be found in Foulis and Bennett (1994).

Theorem 2.2. Let $\mathscr{E}$ be a regular FB-effect algebra; then the structure $\langle \mathscr{E}, \leq, ', 0, 1 \rangle$ is a Kleene poset, i.e., a poset with respect to the partial ordering $\leq$ defined by the $(or - \mathscr{E})$, bounded by the minimum element 0 and the maximum element 1, and equipped with a Kleene (unusual) orthocomplementation $' : \mathscr{E} \mapsto \mathscr{E}$ [which is the mapping defined by the (sa-3)] satisfying, for arbitrary $a, b \in \mathscr{E}$, the conditions

(doc − 1) $a = a''$.
(doc − 2) $a \leq b$ implies $b' \leq a'$.
(re) $a \leq a'$ and $b' \leq b$ imply $a \leq b$ (regularity)

Moreover, we have that

$$a \perp b \qquad \text{iff} \quad a \leq b'$$

Example 2.1. The real unit interval. In this case the partial ordering coincides with the natural ordering of real numbers.

Example 2.2. The standard fuzzy set theory on the reference space U. The partial ordering induced by the SBZ structure is the pointwise ordering on functions:

$$f \leq g \qquad \text{iff} \quad \forall x \in U, \qquad f(x) \leq g(x)$$

Example 2.3. The σ-algebra $\mathscr{B}$(K) of a measurable space K. We have that in this case the partial ordering induced is the standard set-theoretic inclusion:

$$\Delta_1 \leq \Delta_2 \qquad \text{iff} \quad \Delta_1 \subseteq \Delta_2$$

Example 2.4. The unsharp quantum mechanics on the Hilbert space $\mathscr{H}$. The SBZ partial ordering is the phenomenological ordering:

$$F \leq G \quad \text{iff} \quad \forall \psi \in \mathcal{H}, \qquad \langle \psi | F\psi \rangle \leq \langle \psi | G\psi \rangle$$

Example 2.5. The standard quantum logic of a Hilbert space $\mathcal{H}$. The induced SBZ partial ordering is the set theoretical inclusion on subspaces

$$M \leq N \quad \text{iff} \quad M \subseteq N$$

Definition 2.5. Let $\mathcal{E}$ and $\mathcal{F}$ be two Kleene posets. A mapping $\phi: \mathcal{E} \mapsto \mathcal{F}$ is a *K-morphism* iff the following hold:

(i) $\phi(1_{\mathcal{E}}) = 1_{\mathcal{F}}$;

(ii) let $a, b \in \mathcal{E}$; then $a \leq b$ implies $\phi(a) \leq \phi(b)$;

(iii) $\phi(a') = \phi(a)'$.

A *K-epimorphism* is any K-morphism which is onto. A *K-isomorphism* is a K-epimorphism which satisfies the further condition:

(iv) Let $a, b \in \mathcal{E}$; then $\phi(a) \leq \phi(b)$ implies $a \leq b$.

As an immediate consequence of Definition 2.2, we can state the following result.

Proposition 2.3. Let $\mathcal{E}$ and $\mathcal{F}$ be two SBZ-algebras. If $\phi: \mathcal{E} \mapsto \mathcal{F}$ is a po-morphism, then it is a K-morphism.

The notion of morphism in the category of FB effect algebras is straightforwardly obtained once suppress the condition (iv) in Definition 2.1 of an SBZ-algebra morphism. Therefore, any SBZ-algebra morphism is a morphism of FB-effect algebras. Moreover, any morphism between FB-effect algebras is trivially a K-morphism, too. We now consider an example of an FB-effect algebra morphism which is not an SBZ-morphism.

Example 2.6. Let $F \in \mathcal{E}(\mathcal{H})$ be an effect operator on the Hilbert space $\mathcal{H}$; for any nonzero vector $\psi \in \mathcal{H}_0$ let us define the real quantity

$$\phi_F(\psi) := \frac{\langle \psi | F\psi \rangle}{\|\psi\|^2} \in [0, 1]$$

In this way we have defined a mapping $\phi_F: \mathcal{H}_0 \mapsto [0, 1]$, i.e., a fuzzy set in the universe $\mathcal{H}_0$: $\phi_F \in [0, 1]^{\mathcal{H}_0}$. As a consequence the mapping $\phi: \mathcal{E}(\mathcal{H}) \mapsto [0, 1]^{\mathcal{H}_0}$ assigning to any effect operator $F \in \mathcal{E}(\mathcal{H})$ the fuzzy set in the universe $\mathcal{H}_0$, $\phi_F \in [0, 1]^{\mathcal{H}_0}$, is well defined and it is easy to prove that it is an FB-effect algebra morphism. Indeed, $\phi_1 = \underline{1}$ since $\forall \psi \in \mathcal{H}_0$, $\phi_1(\psi) = 1 = \underline{1}(\psi)$; moreover, from $F_1 \perp F_2$, i.e., $0 \leq F_1 + F_2 \leq 1$, it follows that

$$0 \leq \frac{\langle \psi | (F_1 + F_2)\psi \rangle}{\|\psi\|^2} = \frac{\langle \psi | (F_1)\psi \rangle}{\|\psi\|^2} + \frac{\langle \psi | (F_2)\psi \rangle}{\|\psi\|^2} \leq 1$$

that is, $\phi_{F_1} \perp \phi_{F_2}$ and $\phi_{F_1 \oplus F_2} = \phi_{F_1} \oplus \phi_{F_2}$.

This morphism of FB-effect algebras is not an SBZ-algebra morphism since $\phi_{F^\sim} = \phi_{P_{M_0(F)}}$, which is a real unsharp fuzzy set owing to the fact that at every "point" $\psi \in \mathcal{H}_0 \backslash (M_0(F) \cup M_0(F)^\perp)$ it assumes values different from 0 and 1; on the contrary, $(\phi_F)^\sim = \chi_{A_0(\phi_F)}$ is a crisp (sharp) set, characteristic functional of the subset $A_0(\phi_F) = M_0(F) \backslash \{\underline{0}\} = D_0(F)$ of the universe $\mathcal{H}_0$ (incidentally, the certainly-no domain of the fuzzy set ϕ_F is just the certainly-no domain of the effect operator F).

Definition 2.6. A regular FB-effect σ-algebra is a regular FB-effect algebra $\mathscr{E}$ such that the following holds:

(σ) For any sequence $\{a_n\}_{n \in \mathrm{N}} \subseteq \mathscr{E}$ s.t. $\forall n \in \mathrm{N}$, $a_n \leq a_{n+1}$, there exists $\bigvee_{n \in \mathrm{N}} a_n \in \mathscr{E}$.

Remark 2.1. All the examples presented in the preceding section satisfy condition (σ), and thus are examples of regular FB-effect σ-algebras.

Let us notice that in the case of Example 2.4 (the unsharp QM), the existence of the l.u.b of any increasing sequence of effect operators is an immediate corollary of Proposition 1 of Berberian (1966).

In particular we quote the following result, which will be used in the sequel.

Lemma 1. In the SBZ-algebra $[0, 1]$ if $\{a_n\}_{n \in \mathrm{N}} \subseteq [0, 1]$ is an increasing sequence ($\forall n \in \mathrm{N}$, $a_n \leq a_{n+1}$), then $\bigvee_{n \in \mathrm{N}} a_n = \lim a_n$.

Definition 2.7. In a regular FB-effect σ-algebra $\mathscr{E}$ we will say that $\{a_n\}_{n \in \mathrm{N}} \subseteq \mathscr{E}$ is an increasing sequence convergent to $a \in \mathscr{E}$, written $a_n \nearrow a$, iff $\forall n \in \mathrm{N}$, $a_n \leq a_{n+1}$ and $a = \bigvee_{n \in \mathrm{N}} a_n$.

The only structure of FB-"effect" algebra furnishes, at the very least, a notion of "*disjointness*" or "*orthogonality*" $\perp$ for elements of $\mathscr{E}$ and the idea of "*sum*" $\oplus$ for orthogonal elements. These are the right notions in order to introduce inside the FB-effect σ-algebra structure the notions of "*state*" and "*observable*," according to the following definitions which are the abstract version of Definition 1 of Berberian (1966, p. 6) [and also mimics equivalent notions introduced in the context of D-poset structures in Kopka and Chovanec (1994)].

Definition 2.8. Let $\mathscr{E}$ and $\mathscr{F}$ be two regular FB-effect σ-algebras. A mapping $\phi: \mathscr{E} \mapsto \mathscr{F}$ is a σ-morphism iff the following hold:

(i) $\phi(1_{\mathscr{E}}) = 1_{\mathscr{F}}$;

(ii) let $a, b \in \mathscr{E}$ be such that $a \perp b$; then $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$;

(iii) for any $\{a_n\}_{n \in \mathrm{N}} \subseteq \mathscr{E}$ such that $a_n \nearrow a$ it follows that $\phi(a_n) \nearrow \phi(a)$; explicitly

$$\forall n \in \mathrm{N}, \qquad a_n \leq a_{n+1} \quad \text{implies} \quad \phi(\vee a_n) = \vee\phi(a_n)$$

Definition 2.9. Let $\mathscr{E}$ be a regular FB-effect σ-algebra. A (K, $\mathscr{B}$(K))-*observable*, where (K, $\mathscr{B}$(K)) is the measurable space consisting of the value-set K and the σ-algebra $\mathscr{B}$(K) of observable-subsets, is any σ-morphism F: $\mathscr{B}(\mathrm{K}) \mapsto \mathscr{E}$ from the regular FB-effect σ-algebra $\mathscr{B}$(K) (see example 2.3) into $\mathscr{E}$. Explicitly:

(i) $F(\mathrm{K}) = 1$;

(ii) Let $\Delta_1, \Delta_2 \in \mathscr{B}(\mathrm{K})$ be such that $\Delta_1 \cap \Delta_2 = \emptyset$; then $F(\Delta_1) \perp F(\Delta_2)$ and

$$F(\Delta_1 \cup \Delta_2) = F(\Delta_1) \oplus F(\Delta_2)$$

(iii) Let $\{\Delta_n\} \subseteq \mathscr{B}(\mathrm{K})$; then $\forall n \in \mathrm{N}$, $\Delta_n \subseteq \Delta_{n+1}$ implies $F(\cup\Delta_n) = \vee F(\Delta_n)$.

Definition 2.10. Let $\mathscr{E}$ be a regular FB-*effect* σ-algebra. A *state* is any morphism μ: $\mathscr{E} \mapsto [0, 1]$ from $\mathscr{E}$ into the regular FB-effect σ-algebra [0, 1] (see example 2.1). Explicitly:

(i) $\mu(1) = 1$.

(ii) Let $a_1, a_2 \in \mathscr{E}$ be such that $a_1 \perp a_2$; then $\mu(a_1 \oplus a_2) = \mu(a_1) + \mu(a_2) \leq 1$.

(iii) Let $\{a_n\} \subseteq \mathscr{E}$; then $\forall n \in \mathrm{N}$, $a_n \leq a_{n+1}$ implies $\mu(\vee a_n) = \lim \mu(a_n)$.

With a slight modification with respect to Theorem 1 of Kopka and Chovanec (1994) we can state the following result about the "*statistical algorithm*" [Bub (1974) for the adopted terminology].

Proposition 2.4. Let $\mathscr{E}$ be a regular FB-effect σ-algebra. Let μ: $\mathscr{E} \mapsto [0, 1]$ be a state and F: $\mathscr{B}(\mathrm{K}) \mapsto \mathscr{E}$ a (K, $\mathscr{B}$(K))-observable; then the composition pictured by the diagram

$$\begin{tikzcd}
\Delta \in \mathcal{B}(\mathbb{K}) \arrow[r, "F"] \arrow[dr, "(\mu \circ F)"'] & F(\Delta) \in \mathcal{E} \arrow[d, "\mu"] \\
& \mu(F(\Delta)) \in [0,1]
\end{tikzcd}$$

Fig. 1.

defines a mapping $p := (\mu \circ F)$: $\mathscr{B}(\mathrm{K}) \mapsto [0, 1]$ which is a (standard) probability measure.

Proof. Since $(\mu \circ F)$ is finite, nonnegative, additive, and continuous from below at every $\Delta \in \mathscr{B}(\mathrm{K})$, Theorem F of Halmos (1950, p. 39) implies that $(\mu \circ F)$ is a probability measure. ∎

Proposition 2.5. In unsharp quantum mechanics on a Hilbert space of Example 2.4, a (K, $\mathcal{B}$(K))-observable F: $\mathcal{B}(\mathrm{K}) \mapsto \mathcal{E}(\mathcal{H})$ is a (normalized) POV-measure; i.e.,

(pov-1) $F(\mathrm{K}) = 1$;

(pov-2) Let $\Delta_1, \Delta_2 \in \mathcal{B}(\mathrm{K})$ be such that $\Delta_1 \cap \Delta_2 = \emptyset$, then $F(\Delta_1 \cup \Delta_2) = F(\Delta_1) + F(\Delta_2) \leq 1$;

(pov-3) $\forall\{\Delta_n\} \subseteq \mathcal{B}(\mathrm{K})$, $(i \neq j)$, $\Delta_i \cap \Delta_j = \emptyset$ implies

$$F\left(\bigcup_{n=1}^{\infty} \Delta_n\right) = \sum_{n=1}^{\infty} F(\Delta_n)$$

[where the series converges in the weak (also in the ultraweak and strong) operator topology].

Proof. (pov-1) and (pov-2) are nothing but (i) and (ii) of Definition 2.9, specified in the Hilbert space case of Example 2.4.

Let $\{\Delta_n\} \subseteq \mathcal{B}(\mathrm{K})$ be such that $(i \neq j)$ $\Delta_i \cap \Delta_j = \emptyset$, and let us consider the sequence $\Delta_1' = \Delta_1$, $\Delta_2' = \Delta_1 \cup \Delta_2$, and $\forall n \in \mathrm{N}$, $\Delta_n' = \Delta_n \cup \Delta_{n-1}'$; then this is a monotone nondecreasing new sequence such that $\cup\Delta_n' = \cup\Delta_n$, and thus

$$F(\cup\Delta_n) = F(\cup\Delta_n') = \vee F(\Delta_n') \tag{1}$$

Moreover, since $\Delta_n' = \Delta_n \cup \Delta_{n-1}'$ with $\Delta_n \cap \Delta_{n-1}' = \emptyset$, applying (pov-2) for a finite number of steps we get

$$F(\Delta_n') = F(\Delta_1) + \cdots + F(\Delta_n) \tag{2}$$

$\{F(\Delta_n')\}$ is a monotone nondecreasing sequence of effect operators, and thus, by Proposition 1 of Berberian (1966, p. 6) we have that $\exists[\vee F(\Delta_n')]$ such that $\forall\psi \in \mathcal{H}$,

$$\begin{aligned}\langle\psi|[\vee F(\Delta_n')]\psi\rangle &= \lim \langle\psi|F(\Delta_n')\psi\rangle = (2) = \lim \langle\psi| \sum_{i=1}^{n} F(\Delta_i)\psi\rangle \\ &= \langle\psi| \sum_{i=1}^{\infty} F(\Delta_i)\psi\rangle\end{aligned}$$

Applying (1) to this result, we obtain that $\forall\psi \in \mathcal{H}$,

$$\langle\psi|F\left(\bigcup_{n=1}^{\infty} \Delta_n\right)\psi\rangle = \langle\psi| \sum_{n=1}^{\infty} F(\Delta_n)\psi\rangle \quad \blacksquare$$

Proposition 2.6. In unsharp quantum mechanics on a Hilbert space of Example 2.4 the restriction of a state μ: $\mathcal{E}(\mathcal{H}) \mapsto [0, 1]$ (see Definition 2.10) to the orthomodular lattice of projectors $\Pi(\mathcal{H})$ satisfies (s-1) $\mu(1) = 1$;

(s-2) for any orthogonal sequence $\{P_n\} \subseteq \Pi(\mathcal{H})$ of projections [$(i \neq j)$, implies $P_i \perp P_j$] we have

$$\mu\left(\bigvee_{n=1}^{\infty} P_n\right) = \sum_{n=1}^{\infty} \mu(P_n)$$

Proof. From (i) of Definition 2.10 we have that the (s-1) is true. Let now $P_1, P_2 \in \Pi(\mathcal{H})$ be such that $P_1 + P_2 \leq 1$, then from Theorem 2 of Halmos (1951, p. 45) we have that $P_1 \perp P_2$ [i.e., by Theorem 4 of p. 45, $P_1 \circ P_2 = P_2 \circ P_1 = 0$] and $P_1 \vee P_2 = P_1 + P_2$; therefore, from (ii) of Definition 2.10 we get $\mu(P_1 + P_2) = \mu(P_1 \vee P_2) = \mu(P_1) + \mu(P_2) \leq 1$. The extension to a finite number is immediate (once we consider that $[\Sigma_{i=1}^{n} P_i] \perp P_{n+1}$, as a consequence of $P_{n+1} \circ [\Sigma_{i=1}^{n} P_i] = [\Sigma_{i=1}^{n} P_i] \circ P_{n+1} = 0$) and gives

$$P_i \perp P_j \qquad \text{implies} \quad \mu\left(\sum_{i=1}^{n} P_i\right) = \sum_{i=1}^{n} \mu(P_i) \leq 1 \tag{1}$$

Let $\{P_n\} \subseteq \Pi(\mathcal{H})$ be such that $(i \neq j)$, $P_i \perp P_j$, and let us construct the sequence $\hat{P}_1 = P_1$, $\hat{P}_2 = P_1 + P_2$, and $\forall n \in \mathbb{N}$, $\hat{P}_n = P_n + \hat{P}_{n-1}$; then $\forall n \in \mathbb{N}$, $\hat{P}_n \leq \hat{P}_{n+1}$ and

$$\hat{P}_n = \sum_{i=1}^{n} P_i \tag{2}$$

Let us now consider [always owing to Proposition 1 of Berberian (1966, p. 6) applied to the monotone nondecreasing sequence $\{\hat{P}_n\}$]

$$[\vee \hat{P}_n] = \lim \hat{P}_n = (2) = \sum_{n=1}^{\infty} P_n = \vee P_n \tag{3}$$

[where the last equality follows from Theorem 1 of Halmos (1951, p. 49) applied to the orthogonal sequence of projections $\{P_n\}$].

Therefore,

$$\mu\left(\bigvee_{n=1}^{\infty} P_n\right) = (3) = \mu\left(\bigvee_{n=1}^{\infty} \hat{P}_n\right) = \text{(taking into account the Definition 2.10)}$$

$$= \lim \mu(\hat{P}_n) = (2) = \lim \mu\left(\sum_{i=1}^{n} P_i\right) = (1)$$

$$= \lim \sum_{i=1}^{n} \mu(P_i) = \sum_{n=1}^{\infty} \mu(P_n) \quad \blacksquare$$

Conclusion 2. The structure of the SBZ-algebra satisfies the metatheoretical principle (MT_2), which requires that the algebra describing unsharpness permits us to introduce the notion of "oservable" (Definition 2.9) and the notion of "state" (Definition 2.10); these two notions are such that any composition of a state with an observable gives rise to a probability measure.

Moreover, in the SBZ-algebra of Hilbert space unsharp QM, observables are usual (normalized) POV-measures and, using the Gleason Theorem, if the dimension of the Hilbert space is strictly greater than 2, states are in a one-to-one correspondence with density operators.

2.2. Brouwer–Zadeh Poset as a Substructure of SBZ-Algebra: Distinction Between Sharp and Unsharp Elements

We have now to face the problem of distinguishing inside our SBZ-algebraic structure, and in an equational way, sharp elements from unsharp ones. To this end we premise some results in which the second unary operation $\sim: \mathscr{E} \mapsto \mathscr{E}$ of an SBZ-algebra is strongly involved.

Lemma 2. Let $\mathscr{E}$ be an SBZ-algebra and let $a, b \in \mathscr{E}$; then;
(i) $a^{\sim\sim} \oplus a^{\sim} = 1$
(ii) $a^{\sim\sim} = a^{\sim\prime}$
(iii) $a^{\sim} \leq a'$
(iv) $a'^{\sim} \leq a \leq a^{\sim\prime}$
(v) $a \leq b$ implies $b^{\sim} \leq a^{\sim}$. [equivalently, $\exists r$: $a \oplus r = b$ implies $\exists s$: $b^{\sim} \oplus s = a^{\sim}$]
(vi) $\exists r$: $b \oplus r = a$ and $\exists s$: $b \oplus s = a^{\sim}$ imply $b = 0$
(vii) $a \wedge a^{\sim} = 0$
(viii) $a \perp a^{\sim\sim}$ implies $a = 0$

Proof. (i) From property (boc-2) of the definition of SBZ-algebra, applied to the particular case of $a^{\sim}$, we get that $a^{\sim} \perp a^{\sim\sim}$, and thus $a^{\sim} \oplus a^{\sim\sim}$ exists in $\mathscr{E}$. Now, since

$$\exists r = a^{\sim\sim}: \quad a^{\sim} \oplus r = a^{\sim} \oplus a^{\sim\sim} \qquad \text{and}$$
$$\exists s = a^{\sim}: \quad a^{\sim\sim} \oplus s = a^{\sim} \oplus a^{\sim\sim}$$

by (boc-3) we conclude that $a^{\sim} \oplus a^{\sim\sim} = 1$.

(ii) From (i) and (koc-2) we immediately get that $a^{\sim\sim} = (a^{\sim})'$.

(iii) From (boc-2) we have that $a \perp a^{\sim}$, and then from (og-1) we obtain that $a^{\sim} \perp a$, which, owing to (ii), Theorem 2.4 of Foulis and Bennet (1994) implies $a^{\sim} \leq a'$.

(iv) From (iii), written $\forall b$, $b^{\sim} \leq b'$, and applying this inequality to the particular case $b = a'$, we get $\forall a$, $a'^{\sim} \leq a$. Now, from (iii) we have that

$a^{\sim} \le a'$ and so, by (iii) of Theorem 2.4 and (ii) of Lemma 2.3 of Foulis and Bennett (1994) we have that $a = a'' \le a^{\sim\prime}$.

(v) Let $a \le b$ then by (ii) and (iv) we have that $a \le b^{\sim\sim}$, i.e., $\exists r$: $a \oplus r = (b^{\sim})^{\sim}$; applying to this result (boc-1), we have that $\exists s$: $b^{\sim} \oplus s = a^{\sim}$, i.e., by Definition 2.4, $b^{\sim} \le a^{\sim}$.

(vi) Let $a, b, \in \mathscr{E}$ be such that $\exists r$: $b \oplus r = a$ and $\exists s$: $b^{\sim} \oplus s = a^{\sim}$; then by (v) we get that $\exists \hat{r}$: $a^{\sim} \oplus \hat{r} = b^{\sim}$ and $\exists \hat{s}$: $a^{\sim\sim} \oplus \hat{s} = b^{\sim}$; applying the (boc-3) to this result, we get that $b^{\sim} = 1$, from which it follows that $b \le b^{\sim\sim} = b^{\sim\prime} = 0$, concluding that $b = 0$.

(vii) For any $a \in \mathscr{E}$, let b be a lower bound of $\{a, a^{\sim}\}$, i.e., $b \le \{a, a^{\sim}\}$; then $\exists r$: $b \oplus r = a^{\sim}$ and $\exists s$: $b \oplus s = a^{\sim}$; by (vi) we have that $b = 0$, concluding that $a \wedge a^{\sim} = 0$.

(viii) $a \perp a^{\sim\sim}$ implies $a \le a^{\sim\sim\prime} = \text{(ii)} = a^{\sim}$ and thus $a = a \wedge a^{\sim} = \text{(vii)} = 0$. ∎

Theorem 2.3. Let $\langle \mathscr{E}, \perp, \oplus, {}', {}^{\sim}, 0, 1\rangle$ be an SBZ-algebra; then the structure

$$\langle \mathscr{E}, \le, {}', {}^{\sim}, 0, 1\rangle$$

is a bounded [by the minimum element 0 and the maximum element 1] BZ-poset with respect to the partial ordering $\le$ [defined by $(or - \mathscr{E})$ of Definition 2.4] and:

(K-oc) The Kleene (unusual) orthocomplementation $'$: $\mathscr{E} \mapsto \mathscr{E}$ [which is the mapping defined by (iii) of SBZ-algebra] satisfying, for arbitrary a, $b \in \mathscr{E}$, the conditions:

(doc-1) $a = a''$

(doc-2) $a \le b$ implies $b' \le a'$

(re) $a \le a'$ and $b' \le b$ imply $a \le b$ *(regularity)*

(B-oc) The Brouwer (unusual) orthocomplementation $^{\sim}$: $\mathscr{E} \mapsto \mathscr{E}$ [which is the mapping defined by the (iv) of SBZ-algebra] satisfying, whatever be $a, b \in \mathscr{E}$, the following conditions:

(woc-1) $a \le a^{\sim\sim}$

(woc-2) $a \le b$ implies $b^{\sim} \le a^{\sim}$

(woc-3) $a \wedge a^{\sim} = 0$

The two orthocomplementations are linked by the following interconnection rule for every $a \in \Sigma$:

(in) $a^{\sim\prime} = a^{\sim\sim}$

Proof. This result is a consequence of Foulis and Bennett (1994) and the above Lemma 2. ∎

Remark 2.2. From the Kleene orthocomplementation one can induce the binary relation of K-orthogonality on $\mathscr{E}$,

$$a \perp_K b \qquad \text{iff}_{\text{def}} \quad a \leq b'$$

From the Brouwer orthocomplementation one can induce the binary relation of B-orthogonality on $\mathscr{E}$,

$$a \perp_B b \qquad \text{iff}_{\text{def}} \quad a \leq b^{\sim}$$

Let us notice that, as a trivial consequence of the (ii) of Lemma 2,

$$a \perp_B b \qquad \text{implies} \quad a \perp_K b$$

and thus if one introduce a notion of B-morphism similarly to the notion of K-morphism, substituting the only condition (iii) with the condition

(iii-K) Let $a, b \in \mathscr{E}$; if $a \perp_B b$, then $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$

then any B-morphism is also a K-morphism, and in some sense it suffices to study only K-morphisms in order to describe inside SBZ-algebras the standard notions of observable and state.

In any BZ algebra $\mathscr{E}$ it is possible to distinguish the set $\mathscr{E}_s$ of *sharp* (or *exact*) elements, i.e., the collection of those elements from $\mathscr{E}$ which are closed with respect to the Brouwerian orthocomplementation:

$$\mathscr{E}_s := \{\alpha \in \mathscr{E}: \alpha = \alpha^{\sim\sim}\}$$

The elements which are not exact are called *unsharp* (or *fuzzy*).

Lemma 3. Let $\mathscr{E}$ be an SBZ-algebra.
(ix) $\forall \alpha \in \mathscr{E}_s$: if $\alpha \perp \alpha$, then $\alpha = 0$
(x) $\forall \alpha \in \mathscr{E}_s$: $\alpha^{\sim} = \alpha^{\sim\prime}$
(xi) Let $a, b, \in \mathscr{E}$; if one of a, b belongs to $\mathscr{E}_s$, and $a \perp b$, then $a \oplus b$ is the minimal upper bound of a, b, i.e.,

$$\forall c \in \mathscr{E}: \quad \text{if} \quad a, b \leq c \leq a \oplus b, \qquad \text{then} \quad c = a \oplus b$$

Proof. (ix) $\alpha = \alpha^{\sim\sim}$ and $\alpha \perp \alpha$ imply $\alpha \perp \alpha^{\sim\sim}$ and thus, for (viii) Lemma 2 we conclude that $\alpha = 0$.

(x) $\alpha = \alpha^{\sim\sim}$ and (i) of Lemma 2 imply $\alpha \oplus \alpha^{\sim} = 1$, and thus, for the unicity of the orthosupplementation law (sa-3), we have that $\alpha^{\sim} = \alpha'$.

(xi) Let us suppose that $a, b, \leq c \leq a \oplus b$. Then, $\exists d, e \in \mathscr{E}$ s.t. $c = a \oplus d$, $c = b \oplus e$ and $\exists f \in \mathscr{E}$ s.t. $a \oplus b = c \oplus f$. Thus, $a \oplus b = c \oplus f = (a \oplus d) \oplus f = (b \oplus e) \oplus f$. By the cancellation law, $a = e \oplus f$ and $b = d \oplus f$. Thus, $f \leq a$ and $f \leq b$. By hypothesis we have that $a \perp b$ and then, by Theorem 2.2 $a \leq b'$. Again, by hypothesis one of a, b is in $\mathscr{E}_s$; let us suppose $b \in \mathscr{E}_s$ (the proof for $a \in \mathscr{E}_s$ is similar). It turns out that $f \leq b' = b^{\sim}$. Thus, $f \leq b \wedge b^{\sim} = 0$. Therefore, $a \oplus b = c \oplus f = c$. ■

Remark 2.3. Part (x) is a particular result of the BZ poset theory, in which it is possible to show the following stronger result.

Proposition 2.7. In any BZ-poset Σ the following statements are equivalent:

(1) $a^{\sim} = a'$
(2) $a = a'^{\sim}$
(3) $a^{\sim\sim} = a'^{\sim}$
(4) $a = a^{\sim\prime}$
(5) $a = a^{\sim\sim}$

Each of the above conditions implies the following two (mutually equivalent) conditions:

(6a) $a \wedge a' = 0$
(6b) $a \vee a' = 1$

For the proof see Cattaneo and Nisticò (1989)

Theorem 2.4. Let $\mathscr{E}$ be an SBZ-algebra.

(1) The set of sharp elements is nonempty, since $0, 1\ (= 0') \in \mathscr{E}_s$.
(2) $\mathscr{E}_s$ is closed under $\oplus$.
(3) For every $\alpha \in \mathscr{E}_s$, $\alpha' = \alpha^{\sim} \in \mathscr{E}_s$.

Moreover, the structure

$$\langle \mathscr{E}_s, \oplus, ', 0, 1 \rangle$$

is a regular orthoalgebra according to Foulis and Randall (1981) i.e., a regular FB effect algebra in which the zero–one law (sa-4) is replaced by the stronger

(sa-4s) [*Consistency law*] Let $\alpha \in \mathscr{E}_s$, if $\alpha \oplus \alpha$ is defined, then $\alpha = 0$.

Proof. Points (1) and (3) are standard results of BZ poset theory (Cattaneo and Nisticò, 1989). Let us prove point (2). Suppose $a, b \in \mathscr{E}_s$. Then, according to (ii) Lemma 2, $a = a^{\sim\sim} = a'^{\sim}$ and $b = b^{\sim\sim} = b'^{\sim}$. Now, $a, b \le a \oplus b$; therefore, $a, b \le (a \oplus b)'^{\sim}$. Further, by (iv) Lemma 2, $(a \oplus b)'^{\sim} \le (a \oplus b)$. By (xi) of Lemma 3, $(a \oplus b)'^{\sim} = a \oplus b$. Hence, taking into account (ii) of Lemma 2, $(a \oplus b) \in \mathscr{E}_s$. ∎

The following theorem can be obtained as a consequence of both orthoalgebra theory and of BZ poset theory.

Theorem 2.5. Let $\mathscr{E}$ be an SBZ-algebra. Then the structure

$$\langle \mathscr{E}_s, \le, ', 0, 1 \rangle$$

is an orthoposet, sub-BZ poset of $\mathscr{E}$, with respect to the restriction to $\mathscr{E}_s$ of the partial ordering (or-$\mathscr{E}$) of Definition 2.4 and the (standard) orthocomplementation mapping $\alpha \in \mathscr{E}_s \mapsto \alpha' = \alpha^{\sim} \in \mathscr{E}_s$.

Moreover, if the l.u.b. $\vee$ and the g.l.b. $\wedge$ of the pair of sharp elements $\alpha, \beta \in \mathscr{E}_s$ exist in $\mathscr{E}$, then the l.u.b. $\vee_e$ and the g.l.b. $\wedge_e$ of the same pair exist in $\mathscr{E}_s$, and

$$\alpha \vee_e \beta = \alpha \vee \beta \quad \text{and} \quad \alpha \wedge_e \beta = \alpha \wedge \beta$$

Proof. For the proof in the BZ context see Cattaneo and Nisticò (1989). ∎

Making use of the two unusual orthocomplementations, it is possible to define the *weak anti-intuitionistic orthocomplementation:*

$$a \in \mathscr{E} \mapsto a^{\flat} := a'^{\sim\prime} \in \mathscr{E}_s$$

which satisfies the following conditions:

(aoc-1) $a^{\flat\flat} \le a$;

(aoc-2) $a \le b$ implies $b^{\flat} \le a^{\flat}$;

(aoc-3) $a \vee a^{\flat} = 1$.

Trivially, for every $a \in \mathscr{E}$, one has that $a^{\sim} \le a' \le a^{\flat}$. Sharp elements can be equivalently characterized by the anti-intuitionistic negation, since it is easy to prove that $\forall a \in \mathscr{E}$, $a = a^{\sim\sim}$ iff $a = a^{\flat\flat}$. Hence, $\mathscr{E}_s = \{a \in \mathscr{E}: a = a^{\flat\flat}\}$.

In the framework of any BZ-algebra structure it is possible to introduce two unary operators from $\mathscr{E}$ onto $\mathscr{E}_s$ which can be considered as generalized algebraic versions of the "necessity" and the "possibility" connectives of modal logic:

$$a \in \mathscr{E} \mapsto \nu(a) := a'^{\sim} = a^{\flat\flat} \in \mathscr{E}_s \qquad (\textit{necessity})$$

$$a \in \mathscr{E} \mapsto \mu(a) := a^{\sim\prime} = a^{\sim\sim} \in \mathscr{E}_s \qquad (\textit{possibility})$$

In particular the following hold:

(mod-1) The necessity of an element "implies" the element itself, which in it turn "implies" the corresponding possibility

$$\nu(a) \le a \le \mu(a)$$

[$\nu(a) \le \mu(a)$, "*if necessarily a, then possibly a*" is the modal principle D; moreover $\nu(a) \le a$, "*if necessarily a, then a*" is the modal principle T (see Chellas 1980)].

(mod-2) Necessity and possibility are both idempotent

$$\nu(a) = \nu(\nu(a)) \quad \text{and} \quad \mu(a) = \mu(\mu(a))$$

[which is a stronger version of modal principle 4: $\nu(a) \le \nu(\nu(a))$, "*if necessarily a, then necessarily necessarily a*"].

(mod-3) Operators ν and μ act on the exact elements of $\mathscr{E}_s$ as the identity operators:

$$\forall\beta \in \mathscr{E}_s, \qquad \nu(\beta) = \mu(\beta) = \beta$$

[the particular case of $\beta = \mu(a)$ is a stronger version of the modal principle 5: $\mu(a) \leq \nu(\mu(a))$, "*if possibly a, then necessarily possibly a*"].

(mod-4) Necessity and possibility are linked by the expected interconnection rules between modal-like operators

$$\mu(a) = \nu(a')' \quad (\textit{possibility} = \textit{not-necessity-not})$$

$$\nu(a) = \mu(a')' \quad (\textit{necessity}=\textit{not-possibility-not})$$

(mod-5) An interconnection rule involving intuitionistic-like orthocomplementation and modal-like operators can be stated:

$$\nu(a^{\sim}) = \mu(a)^{\sim}$$

[in general, $\mu(a^{\sim}) \neq \nu(a)^{\sim}$]

(mod-6) necessity and possibility are both *monotone*

$$a \leq b \qquad \text{implies} \qquad \nu(a) \leq \nu(b) \quad \text{and} \quad \mu(a) \leq \mu(b)$$

(mod-7) The modal principles of *noncontradiction and excluded-middle* (Moisil 1941, 1941) hold:

$$\nu(a) \wedge \nu(a)' = \mu(a) \wedge \mu(a)' = 0 \quad \text{and} \quad \nu(a) \vee \nu(a)' = \mu(a) \vee \mu(a)' = 1$$

which assume equivalently the weaker form

$$a \wedge \nu(a') = a \wedge \mu(a)' = 0 \quad \text{and} \quad a \vee \nu(a)' = a \vee \mu(a') = 1$$

(mod-8) The two unusual orthocomplementations, both the intuitionistic and the anti-intuitionistic, can be expressed by means of modalities according to the following:

$$a^{\flat} = \nu(a)' = \nu(a)^{\sim} = \nu(a)^{\flat} \qquad (\textit{contingency})$$

$$a^{\sim} = \mu(a)' = \mu(a)^{\sim} = \mu(a)^{\flat} \qquad (\textit{impossibility})$$

Example 2.7. In the real unit interval SBZ algebra [0, 1] the set of exact elements is the two-valued Boolean algebra $\{0, 1\}$.

Example 2.8. Let us consider the SBZ-algebra of all fuzzy sets on the reference space U introduced in Example 2.2, which is also a distributive BZ-lattice with respect to the unusual orthocomplementation mappings: Kleene $f'(x) := (1 - f)(x)$, Brouwer $f^{\sim}(x) := \chi_{A_0(f)}(x)$, and anti-intuitionistic $f^{\flat}(x) := \chi_{A_1(f)_c}(x)$ [where $A_0(f) := \{x \in U: f(x) = 0\}$ is the *impossibility* domain of f, $A_1(f) := \{x \in U: f(x) = 1\}$ the *necessity domain of f*, and so $A_1(f)^c = A_c(f) := \{x \in U \cdot f(x) \neq 1\}$ the *contingency* domain of f].

The set of all "sharp" ("crisp") fuzzy sets $([0, 1]^U)_e = \{f \in [0, 1]^U: f = f^{\sim\sim}\}$ is just the collection of all characteristic functionals on U:

$$([0, 1]^U)_e = \{\chi_A: A \in \mathcal{P}(U)\}$$

For any fuzzy set f the necessity is $\nu(f) = \chi_{A_1(f)}$ and the possibility is $\mu(f) = \chi_{A_p(f)}$ [where $A_p(f) := \{x \in U: f(x) \neq 0\}$ is the *possibility* domain of f].

Example 2.9. In the case of the SBZ-algebra $\mathscr{E}(\mathscr{H})$ of all effect operators of unsharp QM on the Hilbert space $\mathscr{H}$ (see Example 2.4, Section 2) the BZ-substructure is based on the unusual orthocomplementation mappings: Kleene $F' = 1 - F$, Brouwer $F^{\sim} = P_{M_0(F)}$, and anti-intuitionistic $F^{\flat} = P_{M_1(F)^{\perp}}$.

Trivially the set of all "sharp" effect operators $\mathscr{E}(\mathscr{H})_e = \{F \in \mathscr{E}(\mathscr{H}): F = F^{\sim\sim}\}$ is just the set $\Pi(\mathscr{H})$ of all orthogonal projections:

$$\mathscr{E}(\mathscr{H})_e = \Pi(\mathscr{H})$$

The necessity of an effect operator F is $\nu(F) = P_{M_1(F)}$ and the possibility of an effect operator F is $\mu(F) = P_{M_0(F)^{\perp}}$.

2.3. Quantum and Classical SBZ-Algebras of Effects

In Remark 2.2 we have seen that the two orthocomplementations of the BZ poset structure induced from any SBZ algebra $\mathscr{E}$ give rise to two orthogonality relations $\perp_K$ and $\perp_B$. Now, a link between these two orthogonalities is given by the possibility according to the following result.

Proposition 2.8. Let $a, b \in \mathscr{E}$. Then

$$a \perp_{\mathrm{B}} b \quad \text{iff} \quad \mu(a) \perp_{\mathrm{K}} \mu(b)$$

Proof. $a \perp_{\mathrm{B}} b$ iff $a \leq b^{\sim}$; which implies $\mu(b) = b^{\sim\sim} \leq a^{\sim} = (a^{\sim\prime})' = \mu(a)'$, i.e., $\mu(a) \perp_{\mathrm{K}} \mu(b)$.

Conversely, $\mu(a) \perp_{\mathrm{K}} \mu(b)$ iff $b^{\sim\sim} \leq a^{\sim}$, which implies $a \leq a^{\sim\sim} \leq b^{\sim\sim\sim} = b^{\sim}$ i.e., $a \perp_{\mathrm{B}} b$. ∎

Definition 2.11. A quantum SBF algebra is any SBZ algebra $\mathscr{E}$ satisfying the following B-coherence law:

(q) For any triple $a, b, c \in \mathscr{E}$ of pairwise B-orthogonal elements, written $\{a, b, c\} \perp_B$, there exists the sum $a \oplus b \oplus c \in \mathscr{E}$.

Remark 2.4. Let us stress that in the above B-coherence law what is involved is the B-orthogonality relation, differently from Foulis and Bennett

(1994) where one has to do with a coherence law involving the Kleene orthocomplementation. Indeed, the Foulis-Bennett coherence law can also be called a K- coherence law, since it can be expressed as:

- For any triple $a, b, c \in \mathscr{E}$ of pairwise K-orthogonal elements, written $\{a, b, c\} \perp_K$, there exists the sum $a \oplus b \oplus c \in \mathscr{E}$.

Let us recall Theorem 5.3 of Foulis and Bennett (1994): An FB-effect algebra is an orthomodular poset iff it satisfies the *K-coherence law.* Differently, if an SBZ alegbra satisfies the B-coherence law, then we cannot state that it is an orthomodular poset (see the example below of the SBZ algebra of effect operators on a Hilbert space).

Proposition 2.9. Let $\mathscr{E}$ be a q-SBZ algebra. Then, the set $\mathscr{E}_s$ of all sharp elements is an orthomodular poset.

Proof. From the fact that on the orthoposet $\mathscr{E}_s$ of all exact elements the Kleene and the Brouwer orthocomplementations collapse into a unique standard orthocomplementation, $\forall \alpha \in \mathscr{E}_s$, $\alpha' = \alpha^{\sim}$, we have that $\forall \alpha, \beta \in \mathscr{E}_s$, $\alpha \perp_K \beta$ iff $\alpha \perp_B \beta$. Therefore, for any triple $\alpha, \beta, \gamma \in \mathscr{E}_s$,condition $\{\alpha, \beta, \gamma\} \perp_B$ is equivalent to $\{\alpha, \beta, \gamma\} \perp_K$, and owing to the q-axiom, this implies the existence of $\alpha \oplus \beta \oplus \gamma$ which is an element of $\mathscr{E}_s$ by condition (2) of Theorem 2.4. Then, applying Theorem 5.3 of Foulis and Bennett (1994) to the orthoalgebra (which is an FB-effect algebra, too) $\mathscr{E}_s$ we conclude the thesis. ■

Proposition 2.10. The SBZ algebra $\mathscr{E}(\mathscr{H})$ of all effect operators on a Hilbert space $\mathscr{H}$ satisfies the B-coherence law.

Proof. Let F, G, T be three effect operators which are pairwise B-orthogonal; this means that, according to Proposition 2.8, the three projectors $P_{\mathrm{Ran}(F)}$, $P_{\mathrm{Ran}(G)}$, $P_{\mathrm{Ran}(T)}$ are pairwise orthogonal with respect to the standard orthocomplementation on $\Pi(\mathscr{H})$. As a standard result on Hilbert space theory, we have that $P_{\mathrm{Ran}(F)} + P_{\mathrm{Ran}(G)} + P_{\mathrm{Ran}(T)}$ is a projector, in particular $P_{\mathrm{Ran}(F)} + P_{\mathrm{Ran}(G)} + P_{\mathrm{Ran}(T)} \leq \mathrm{I}$. Since for any effect operator $F \in \mathscr{E}(\mathscr{H})$ the following inclusion holds, $F \leq P_{\mathrm{Ran}(F)}$, we have that $0 \leq F + G + T \leq P_{\mathrm{Ran}(F)} + P_{\mathrm{Ran}(G)} + P_{\mathrm{Ran}(T)} \leq \mathrm{I}$, and so $F + G + T = F \oplus G \oplus T \in \mathscr{F}(\mathscr{H})$. ■

Remark 2.5. As recalled in Section 1, and consistently with the above result, it is well known that the set $\Pi(\mathscr{H})$ of all projectors on a Hilbert space, as collection of all sharp elements of the SBZ algebra $\mathscr{E}(\mathscr{H})$ of all effect operators $[(\mathscr{E}(\mathscr{H}))_s = \Pi(\mathscr{H})]$, is an orthomodular (atomic, complete) lattice.

Definition 2.12. A classical (c) SBZ algebra is any q-SBZ algebra $\mathscr{E}$ satisfying the following B-compatibility law:

(c) For any pair of elements $a, b, \in \mathscr{E}$, there exists a_1, b_1, $c \in \mathscr{E}$ such that $a_2 \perp_B c$, $a_1 \perp_B (a_2 \oplus c)$, with $a = a_1 \oplus c$ and $b = b_1 \oplus c$.

Making use of the same technique used in proving Proposition 2.9 and the results of Foulis *et al.* (1992) (or also Foulis and Bennett, 1994), the following is straightforward.

Proposition 2.11. Let $\mathscr{C}$ be a c-SBZ algebra. Then, the set of $\mathscr{C}_s$, of all sharp elements is a Boolean algebra.

Proposition 2.12. The SBZ algebra $[0, 1]^U$ of all fuzzy sets on the universe U is a c-SBZ.

Proof. Trivial consequence of the fact that two fuzzy sets f, g are B-orthogonal iff their possibility domains (also supports) are disjoint. ■

Remark 2.6. In the SBA algebra $[0, 1]^U$ of all fuzzy sets, as shown in Example 2.8, the set $\{0, 1\}^U$ of all characteristic functionals of subsets of U, as collection of all sharp (crisp) elements, is a Boolean algebra isomorphic to the power set $\mathcal{P}(U)$ of U.

Example 2.10. The case of fuzzy sets on the universe U gives an example of the fact that B-coherence can be satisfied, whereas in general K-coherence is not true.

Indeed, from the fact that

$$f, g \in [0, 1]^U, \qquad f \perp_B g \quad \text{iff} \quad \Delta_p(f) \cap \Delta_p(g) = \emptyset$$

we trivially obtain that from any triple f, g, $h \in [0, 1]^U$ of pairwise B-orthogonal elements the sum $f + g + h$ is a fuzzy set.

But for instance in the universe R the triple

$$f = \frac{1}{2}\chi_{[-1,1]}, \qquad g = \chi_{[-2,-1)} + \frac{1}{2}\chi_{[-1,1]}, \qquad h = \frac{1}{2}\chi_{[-1,1]} + \chi_{(1,2]}$$

consists of pairwise K-elements and $f + g + h = 1/2\,\chi_{[-2,-1]} + 3/2\,\chi_{[-1,1]}$ $1/2\,\chi_{[1,2]} \notin [0, 1]^{\mathbf{R}}$.

Similar pathological examples can be found with respect to a K-version of the compatibility law applied to fuzzy sets which are not crisp.

2.4. Modal-Like Operators and Rough Approximation Mapping in BZ Structures

From any elements $a \in \mathscr{C}$ the associated necessity $\nu(a)$ and possibility $\mu(a)$ can be considered respectively as:

(1). the *lower.* (*or inner*) sharp approximation of a (approximation of a from the bottom by sharp elements), since one can prove that

$$\nu(a) = \vee\{\beta \in \mathscr{C}_e : \beta \leq a\}$$

$$\nu(a) \in \{\beta \in \mathscr{C}_e : \beta \leq a\}$$

(2) the *upper* (or *outer*) sharp approximation of a (approximation of a from the top by sharp elements), since can prove that

$$\mu(a) = \wedge\{\gamma \in \mathscr{E}_e : a \leq \gamma\}$$

$$\mu(a) \in \{\gamma \in \mathscr{E}_e : a \leq \gamma\}$$

$a \equiv b$ iff $\nu(a) = \nu(b)$ is an equivalence relation on $\mathscr{E}$. Any equivalence class modulo $\equiv$ is called a *property*, we shall denote by $pr(a)$ the equivalence class generated by the element $a \in \mathscr{E}$ and any element $\hat{a} \in pr(a)$ is said to be a *representative* of property $pr(a)$. The following hold:

(i) the necessity ν(a) belongs to property $pr(a)$ and is the *unique exact representative* of this property [hence, all the other elements from the same property are its *fuzzy representatives*].

(ii) $\nu(a)$ minimizes the property $pr(a)$: $\nu(a) = \wedge pr(a)$.

(iii) $\nu(a)$ is the best "sharp" approximation from the bottom of every fuzzy representative $\hat{a}$ of property $pr(a)$: $\forall \hat{a} \in pr(a)$, $\nu(a) = \vee\{\hat{x} \in \mathscr{E}_e: \hat{x} \leq \hat{a}\}$.

Therefore, any property can be sharply identified with its unique exact representative:

$$pr(a) \leftrightarrow \nu(a) \qquad \text{(property)} \leftrightarrow \text{(necessity)}$$

$a \equiv_0 b$ iff $\mu(a) = \mu(b)$ is an equivalence relation on $\mathscr{E}$. The equivalence class generated by $a \in \mathscr{E}$ is denoted by $pr_0(a)$ and called *noperty*. $\mu(a)$ is the unique exact representative of noperty $pr_0(a)$, all other elements from $pr_0(a)$ are fuzzy representatives. Therefore, $pr_0(a)$ can be identified with its unique exact representative:

$$pr_0(a) \leftrightarrow \mu(a) \qquad \text{(noperty)} \leftrightarrow \text{(possibility)}$$

From another point of view, $pr_0(a)$ can also be identified with the exact element $\mu(a)'$, which, of course, does not belong to this class:

$$pr_0(a) \leftrightarrow \mu(a)' = a^{\sim} \qquad \text{(noperty)} \leftrightarrow \text{(impossibility)}$$

The *rough* approximation of any $a \in \mathscr{E}$ by sharp elements is the "necessity–possibility" ordered pair

$$r(a) := (\nu(a), \mu(a)) \qquad [\text{with } \nu(a) \leq \mu(a)]$$

pictured by the following diagram:

$$\begin{tikzcd}
& a \in \mathcal{E} \arrow[dl, "\nu"'] \arrow[dd, "r"] \arrow[dr, "\mu"] & \\
\nu(a) \in \mathcal{E}_e \arrow[dr] & & \mu(a) \in \mathcal{E}_e \arrow[dl] \\
& (\nu(a), \mu(a)) &
\end{tikzcd}$$

Fig. 2.

The above "necessity–possibility" pair can be identified with the "necessity-impossibility" orthopair

$$r_{BZ}(a) := (\nu(a), \mu(a)') \qquad [\text{with } \nu(a) \perp \mu(a)']$$

pictured by the diagram:

$$\begin{tikzcd}
& & a \in \mathcal{E} \arrow[dl, "\nu"'] \arrow[dd, "r_{BZ}"] \arrow[dr, "\mu(\cdot)'"] & & \\
pr(a) \arrow[r, leftrightarrow] & \nu(a) \in \mathcal{E}_e \arrow[dr] & & \mu(a)' \in \mathcal{E}_e \arrow[dl] \arrow[r, leftrightarrow] & pr_0(a) \\
& & (\nu(a), \mu(a)') & &
\end{tikzcd}$$

Fig. 3.

Example 2.11. Rough approximations in fuzzy set theory.

Let us recall that the necessity of a fuzzy set f is $\nu(f) = \chi_{A_1(f)}$ and the possibility is $\mu(f) = \chi_{A_p(f)}$; thus, the impossibility is given by $\mu(f)' = f^{\sim} = \chi_{A_0(f)}$. Hence, two fuzzy sets define a property iff they have the same certainly-yes domain; in this way, to every property of fuzzy sets we can associate the certainly-yes domain of any of its elements. This property is interpreted as: "the point belongs with certainty to the subset $A_1(f)$ of U" and thus any property is exactly represented by the characteristic functional $\chi_{A_1(f)}$.

Similarly, two fuzzy sets define the same noperty iff they have the same certainly-no domain. To every noperty we can associate the unique subset $A_0(f)$ of U which represents the noperty: "the point does not belong with certainty to the set $A_0(f)$" and this noperty is exactly represented by the characteristic functional $\chi_{Ao(f)}$.

The rough approximation of a fuzzy set f by sharp sets is the necessity–possibility pair, identified with the pair of ordered subsets of U:

$$r(f) = (\chi_{A_1(f)}, \chi_{A_p(f)}) \equiv (A_1(f), A_p(f)) \qquad [\text{with } A_1(f) \subseteq A_p(f)]$$

From another point of view, one can also construct the rough approximation of a fuzzy set as the necessity–impossibility pair, identified with the pair of disjoint subsets of U:

$$r_{BZ}(f) = (\chi_{A_1(f)}, \chi_{A_0(f)}) \equiv (A_1(f), A_0(f)) \qquad [\text{with } A_1(f) \cap A_0(f) = \emptyset]$$

All this can be summarized by the following diagram:

$$\begin{tikzcd}
 & & f \in [0,1]^{\mathcal{U}} \arrow[dll, "\nu"'] \arrow[dd, "r_{BZ}"] \arrow[drr, "\mu(\cdot)'"] & & \\
\chi_{A_1(f)} \arrow[r, leftrightarrow] & A_1(f) \arrow[dr] & & A_0(f) \arrow[dl] \arrow[r, leftrightarrow] & \chi_{A_0(f)} \\
 & & (A_1(f), A_0(f)) & &
\end{tikzcd}$$

Fig. 4.

Example 2.12. Rough approximation in unsharp quantum mechanics.

The necessity of an effect operator F is $\nu(F) = P_{M_1(F)}$ and the possibility of an effect operator F is $\mu(F) = P_{M_0(F)^\perp}$ and so the associated impossibility $\mu(F)' = P_{M_0(F)}$. Therefore, two effects define a property iff they have the same certainly-yes subspace, collection of all preparations ψ in which the two effects occur with certainty (probability one).

The possibility of an effect operator F is $\mu(F) = P_{M_0(F)\perp}$; thus two effects belong to the same if they have a common certainly-no subspace, collection of all preparations in which the two effects does not occur with certainty (probability zero).

The corresponding rough approximation of an effect operator F is the necessity–possibility pair, identified with the ordered pair of subspaces of $\mathcal{H}$:

$$r(F) = (P_{M_1(F)}, P_{M_0(F)_\perp}) \equiv (M_1(F), M_0(F)^\perp) \qquad [\text{with } M_1(F) \subseteq M_0(F)^\perp]$$

One can also give a necessity–impossibility rough approximation of the given effect, identified with the ortho pair of subspaces

$$r_{BZ}(F) = (P_{M_1(F)}, P_{M_0(F)}) \equiv (M_1(F), M_0(F)) \qquad [\text{with } M_1(F) \perp M_0(F)]$$

All this is summarized in the following diagram:

$$\begin{tikzcd}
 & & F \in \mathcal{E}(\mathcal{H}) \arrow[dll, "\nu"'] \arrow[drr, "\mu(\cdot)'"] \arrow[dd, "r_{BZ}"] & & \\
P_{M_1(F)} \arrow[r, leftrightarrow] & M_1(F) \arrow[dr] & & M_0(F) \arrow[r, leftrightarrow] \arrow[dl] & P_{M_0(F)} \\
 & & (M_1(F), M_0(F)) & &
\end{tikzcd}$$

Fig. 5.

As a conclusion we can claim that the abstract SBZ-algebraic structure introduced in this work, besides the metatheoretical principles (MT_1) and (MT_2) previously verified, satisfies also the remaining (MT_3) and (MT_4) principles. Indeed, we can state the further following result:

Conclusion 3. In an SBZ-algebra the set of all "sharp" elements is the equational class of all Brouwerian-closed elements: $a \in \mathscr{E}$ such that $a = a^{\sim\sim}$ (equivalently, anti-Brouwerian-closed elements: $a \in \mathscr{E}$ such that $a = a^{\flat\flat}$).

In the unsharp quantum mechanics of effect operators in a Hilbert space the SBZ-"sharp" elements are the orthogonal projections.

Conclusion 4. The BZ-substructure permits us to associate to any element $a \in \mathscr{E}$ of an SBZ-algebra the rough approximation $r(a) = (\nu(a), \mu(a))$ by the necessity of a (the best "sharp" approximation of a from the bottom) and the possibility of a (the best "sharp" approximation of a from the top).

In the Hilbert space case, the BZ-rough approximation of an effect operator F, as a "necessity–impossibility" pair, is identified with the ortho-pair of subspaces ($M_1(F)$, $M_0(F)$) consisting of the certainly-yes domain [collection of all preparations in which the effect occurs with certainty (i.e., probability one)] and of the certainly-no domain [collection of all preparations in which the effect does not occur with certainty (i.e., probability zero)].

3. THE SBZ-LIKE ALGEBRA OF CLASSICAL AND QUANTUM UNSHARP PROPOSITIONAL LOGICS

In Section 1 we set out four metatheoretical principles to which possible algebraic structures describing unsharpness in quantum physics must conform. In particular (MT_1)–(MT_3) require to have as concrete mathematical model effects operators of Hilbert spaces. On the other hand, we have seen that the identification in the category of orthomodular lattices between orthogonal projections and subspaces of a Hilbert space is broken up in the case of effect operators. To be precise, we can only state the (surjective) "rough approximation" mapping from the family of all effects operators onto the family of all ordered pairs of subspaces; this mapping associates with any effect operator $F \in \mathscr{E}(\mathscr{H})$ the quantum "proposition" $r(F) = (M_1(F), M_0F^{\perp})$, with $M_1(F \subseteq M_0(F)^{\perp}$. It will be interesting to see if also the set of all quantum propositions can be equipped with a structure of SBZ-like algebra.

Let us discuss now two interesting examples.

Example 1: The Unsharp "Classical Logic" of a Measurable Space $(\mathbb{K}, \mathscr{B}(\mathbb{K}))$

Let us consider $\mathscr{B}(\mathbb{K}) \times \mathscr{B}(\mathbb{K})) \subseteq = \{(A_1, A_p): A_1, A_p \in \mathscr{B}(\mathbb{K}), A_1 \subseteq A_p\}$. Then this set can be equipped with an SBZ-like structure

$$\langle(\mathscr{B}(\mathbb{K}) \times \mathscr{B}(\mathbb{K}))_{\subseteq}, \perp, \oplus, ', {}^{\sim}, (\emptyset, \emptyset), (\mathbb{K}, \mathbb{K})\rangle$$

with respect to:

(1) the *orthogonality* relation: let $(A_1, A_p), (B_1, B_p) \in (\mathscr{B}(\mathbb{K}) \times \mathscr{B}(\mathbb{K}))_{\subseteq}$; then

$$(A_1, A_p) \perp (B_1, B_p) \quad \text{iff} \quad A_1 \subseteq (B_p)^c \text{ and } B_1 \subseteq (A_p)^c$$

[note that $(A_1, A_p) \perp (A_1, A_p)$ iff $A_1 = \emptyset$].

(2) the *partial sum* operation: let $(A_1, A_p), (B_1, B_p) \in (\mathscr{B}(\mathbb{K}) \times \mathscr{B}(\mathbb{K}))_{\subseteq}$, with $(A_1, A_p) \perp (B_1, B_p)$; then

$$(A_1, A_p) \oplus (B_1, B_p) := (A_1 \cup B_1, A_p \cup B_p)$$

(3) The K-orthocomplementation: let $(A_1, A_p) \in (\mathscr{B}(\mathbb{K}) \times \mathscr{B}(\mathbb{K}))_{\subseteq}$; then

$$(A_1, A_p)' := ((A_p)^c, (A_1)^c)$$

(4) The B-orthocomplementation: let $(A_1, A_p) \in (\mathscr{B}(\mathbb{K}) \times \mathscr{B}(\mathbb{K}))_{\subseteq}$; then

$$(A_1, A_p)^{\sim} := ((A_p)^c, (A_p)^c)$$

Example 2: The Unsharp "Quantum Logic" of a Hilbert Space $\mathscr{H}$

Let us consider $(\mathcal{M}(\mathscr{H}) \times \mathcal{M}(\mathscr{H}))_{\subseteq} := \{(M_1, M_p): M_1, M_p \in \mathcal{M}(\mathscr{H}), M_1 \subseteq M_p\}$. Then this set can be equipped with an SBZ-like structure:

$$\langle(\mathcal{M}(\mathscr{H}) \times \mathcal{M}(\mathscr{H}))_{\subseteq}, \perp, \oplus, ', {}^{\sim}, (\{\underline{0}\}, \{\underline{0}\}), (\mathscr{H}, \mathscr{H})\rangle$$

with respect to:

(1) The *orthogonality* relation: let $(M_1, M_p), (N_1, N_p) \in (\mathcal{M}(\mathscr{H}) \times \mathcal{M}(\mathscr{H}))_{\subseteq}$; then

$$(M_1, M_p) \perp (N_1, N_p) \qquad \text{iff} \quad M_1 \subseteq (N_p)^{\perp} \qquad \text{and} \quad N_1 \subseteq (M_p)^{\perp}$$

$[(M_1, M_p) \perp (M_1\ M_p)$ iff M_1 is the trivial subspace $\{\underline{0}\}]$.

(2) The partial sum operation: let $(M_1, M_p), (N_1, N_p) \in (\mathcal{M}(\mathcal{H}) \times \mathcal{M}(\mathcal{H}))_{\subseteq}$ be such that $(M_1, M_p) \perp (N_1, N_p)$; then

$$(M_1, M_p) \oplus (N_1, N_p) := (M_1 \vee N_1, M_p \vee N_p)$$

(3) The K-orthocomplementation: let $(M_1, M_p) \in (\mathcal{M}(\mathcal{H}) \times \mathcal{M}(\mathcal{H}))_{\subseteq}$; then

$$(M_1, M_p)' := ((M_p)^{\perp}, (M_1)^{\perp})$$

(4) The B-orthocomplementation: let $(M_1\ M_p) \in (\mathcal{M}(\mathcal{H}) \times \mathcal{M}(\mathcal{H}))_{\subseteq}$; then

$$(M_1, M_p)^{\sim} := ((M_p)^{\perp}, (M_p)^{\perp})$$

The above algebraic structures are SBZ-like, since we can prove the following:

Theorem 3.1 Let a, b be two elements of either $(\mathcal{B}(\mathrm{K}) \times \mathcal{B}(\mathrm{K}))_{\subseteq}$ or $(\mathcal{M}(\mathcal{H}) \times \mathcal{M}(\mathcal{H}))_{\subseteq}$. Then the following hold [where we set $0 := (\emptyset, \emptyset)$ and $1 := (\mathrm{K}, \mathrm{K})$ in the classical case, and $0 := (\{\underline{0}\}, \{\underline{0}\})$ and $1 := (\mathcal{H}, \mathcal{H})$ in the quantum one].

(og-1) {*Symmetry law*}

$$a \perp b \quad \text{implies} \quad b \perp a$$

(og-2) {*Regularity law*}

$$a \perp a \quad \text{and} \quad b \perp b \quad \text{imply} \quad a \perp b$$

(og-3) {*Zero–One law*}

$$1 \perp a \quad \text{implies} \quad a = 0$$

Moreover, the partial sum operation is such that:

(sa-1) [*Commutative law*] If $a \perp b$, then

$$a \oplus b = b \oplus a$$

(sa-2) [*Associative law*] If $a \perp b$ and $(a \oplus b) \perp c$, then $b \perp c$, $a \perp (b \oplus c)$, and

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c$$

Furthermore, the *K*-orthocomplementation satisfies:

(koc-1w) [*Weak K-orthosupplementation law*]

$$a \perp a' \quad \text{and} \quad a = a''$$

(koc-2) [*K-Uniqueness law*]

$$a \perp b \quad \text{and} \quad a \oplus b = 1 \quad \text{imply} \quad b = a'$$

Lastly, the B-orthocomplementation satisfies

(boc-1w) [*B-Weak symmetry law*]

$$\exists r\colon\ a \oplus r = b^{\sim\sim} \quad \text{implies} \quad \exists s\colon\ b^{\sim} \oplus s = a^{\sim}$$

(boc-2) [*B-Orthogonality law*]

$$a \perp a^{\sim}$$

(boc-3) [*B-Noncontradiction law*]

$$\exists r\colon\ a^{\sim} \oplus r = c \quad \text{and} \quad \exists s\colon\ a^{\sim\sim} \oplus s = c \quad \text{imply} \quad c = 1$$

Proof. We give a proof of the classical case only, where $a = (A_1, A_p)$, $b = (B_1, B_p)$. The quantum one is similar.

(og-1) is trivial. Part (og-2) follows from $A_1 = \emptyset \subseteq (B_p)^c$ and $B_1 = \emptyset \subseteq (A_p)^c$. Let now $(U, U) \perp (A_1, A_p)$; then $U \subseteq (A_p)^c$ and $A_1 \subseteq U^c$ imply $A_1 = A_p = \emptyset$. The commutative and associative laws are trivially verified.

From $A_1 \subseteq (A_1^c)^c$ and $B_1 \subseteq (B_1^c)^c$ we obtain $(A_1, A_p) \perp ((A_p)^c, (A_1)^c)$; moreover, $(A_1, A_p)'' = ((A_p)^c, (A_1)^c)' = (A_1, A_p)$, which is the (koc-1w).

For the K-uniqueness law, we prove that $(A_1, A_p) \perp (B_1, B_p)$ and $(A_1, A_p) \oplus (B_1, B_p) = (U, U)$ imply necessarily that $(A_1, A_p) = (A_1, A_1)$ and $(B_1, B_p) = ((A_1)^c, (A_1)^c) = (A_1, A_1)^c$.

Indeed, the hypothesis can be restated as $A_1 \subseteq (B_p)^c$ and $B_1 \subseteq (A)^c$, and $A_1 \cup B_1 = A_p \cup B_p = U$. The first inclusion can be extended to $A_1 \subseteq (B_p)^c \subseteq (B_1)^c$, i.e., $A_1 \cap B_1 = \emptyset$, and taking into account the later identity $A_1 \cup B_1 = U$, we obtain that $(A_1)^c = B_1$; moreover, always the first inclusion implies $B_1 \subseteq B_p \subseteq (A_1)^c$, and thus we conclude that $B_1 = B_p = (A_1)^c$. Owing to this result, condition $(A_1, A_p) \perp (B_1, B_p)$ turns out to be $(A_1, A_p) \perp (A_1)^c$, $(A_1)^c)$, which in particular implies that $(A_1)^c \subseteq (A_p)^c$, i.e., $A_p \subseteq A_1$, from which $A_1 = A_p$ follows.

If $\exists (R_1, R_p)\colon (A_1\ A_p) \oplus (R_1, R_p) = (B_1, B_p)^{\sim\sim} = (B_p, B_p)$, then in particular $A_1 \subseteq (R_p)^c$, which implies

$$R_p \subseteq (A_1)^c \tag{1}$$

On the other hand, $A_p \cup R_p = B_p$ implies

$$R_p \subseteq B_p \tag{2}$$

Lastly, from $R_1 \subseteq (A_p)^c \subseteq (A_1)^c$ we get $R_1 \cap A_1 = \emptyset$, and taking into account that $A_1 \cup R_1 = B_p$; we obtain

$$R_1 = B_p \cap (A_1)^c \tag{3}$$

In conclusion, $R_a \subseteq R_p = (1) = R_p \cap (A_1)^c \subseteq (2) \subseteq B_p \cap (A_1)^c = (3) =$

R_1, i.e., $R_1 = R_p$. Under this result, the hypothesis assumes the form $(A_1, A_p) \oplus (R_1, R_1) = (B_p, B_p)$, where $A_1 \subseteq (R_1)^c$ and $R_1 \subseteq (A_p)^c$, i.e., $A_1 \cap R_1 = A_p \cap R_a = \emptyset$; moreover, it must be $A_1 \cup R_1 = A_p \cup R_1 = B_p$. From this result it follows that necessarily $A_1 = A_p$; and thus the hypothesis becomes $(A_1, A_1) \oplus (R_1, R_1) = (B_p, B_p)$, with $R_t \subseteq (A_1)^c$ (i.e., $A_1 \cap R_1 = \emptyset$) and $A_1 \cup R_1 = B_p$, which imply $A_1 = (R_1)^c \cap B_p$, i.e.,

$$(A_1)^c = R_1 \cap (B_p)^c \tag{4}$$

From $A_1 \cup R_1 = B_p$ it follows the $R_1 \subseteq B_p$, i.e., $(B_p)^c \subseteq (R_1)^c$, and so

$$((B_p)^c, (B_p)^c) \perp (R_1, R_1) \tag{5}$$

Making use of (4) and (5) we conclude that $((B_p)^c, (B_p)^c) \oplus (R_1, R_1) = ((A_1)^c, (A_1)^c)$.

$(A_1, A_p)^\sim = ((A_p)^c, (A_p)^c)$ and $A_1 \subseteq A_p = ((A_p)^c)^c$ and $(A_p)^c \subseteq (A_1)^c$ trivially imply $(A_1, A_p) \perp (A_1, A_p)^\sim$.

$(A_1, A_p)^\sim \oplus (R_1, R_p) = (C_1, C_p)$ and $(A_1, A_p)^{\sim\sim} \oplus (S_1, S_2) = (C_1, C_2)$ mean $((A_p)^c, (A_p)^c) \oplus (R_1, R_p) = (C_1, C_2)$ and $(A_p, A_p) \oplus (S_1, S_2)$. Therefore, $C_1 = (A_p)^c \cup R_1 = A_p \cup S_1$, with $R_1 \cap (A_p)^c = S_1 \cap A_1 = \emptyset$; this implies $A_p = R_1$ and $(A_p)^c = S_1$ and thus $C_1 = A_p \cup S_1 = A_p \cup (A_p)^c = U$. So it must be also that $C_p = U$, i.e., $(C_1, C_p) = (U, U)$.

Remark 3.1. From the above Theorem we get that the SBZ-like structure now introduced differs from the SBZ-algebra considered in Section 2 in the properties (koc-1) and (boc-2). As stated in (Foulis and Bennett (1994) (koc-1) implies (koc-2w), and thus the latter is a generalization of the former.

In particular, let us notice that for instance in the classical case

$$(A_1, A_p) \oplus (A_1, A_p)' = ((A_p \setminus A_1)^c, U)$$

which is equal to (U, U), iff $A_1 = A_p$. Pairs of the kind (A, A) [i.e., $A_1 = A_p$] are just the ones for which $(A_1, A_p) = (A_1, A_p)^{\sim\sim}$ (i.e., the sharp elements).

At a first glance, it seems that no relation may be stated between (boc-1) and (boc-1w); but this presently is an open problem, which can be an argument for a forthcoming deep study about these new SBZ-like structures describing unsharp algebras for both the classical and the quantum propositional logics.

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REFERENCES

Berberian, S. K. (1966). *Notes on Spectral Theory*, Van Nostrand, Princeton, New Jersey.

Bub, J. (1974). *The Interpretation of Quantum Mechanics*, D. Reidel, Dordrecht.

Cattaneo, G., (1976). Mathematical foundations of roughness and fuzziness in *The Fourth International Workshop on Rough Sets, Fuzzy Sets, and Machine Discovery—RSFD96*, S. Tsumoto *et al.*, eds., Tokyo Japan, *241–247.*

Cattaneo G., and Marino G. (1988). Non-usual orthocomplementations on partially ordered sets and fuzziness, *Fuzzy Sets and Systems* **25**, 107–123.

Cattaneo, G. and Nističo, G. (1989) Brouwer-Zadeh posets and three valued Lukasiewicz posets, *Fuzzy Sets and Systems* **33**, 165–190.

Chellas, B. F. (1980). *Modal Logic, An Introduction*, Cambridge University Press, Cambridge, 1980.

Cignoli, R., and Monteiro, A. (1965). Boolean elements in Lukasiewicz algebras. II, *Proceeding Japan Academy,* **41**, 676–680.

Cignoli, R. (1970). *Moisil Algebras*, Instituto de Matematica, Universidad Nacional del sur, Bahia Blanca, Argentina.

Davies, E. B., (1976). *Quantum Theory of Open Systems*, Academic Press, London.

Foulis, D. J., and Bennett M. K. (1994). Effect algebras and unsharp quantum logics, *Foundations of Physics,* **24**, 1331–1352.

Foulis, D. J., and Randall, C. (1981). Empirical Logic and tensor product, in *Interpretation and Foundations of Quantum Mechanics*, H. Neumann, ed., Wissenschaftsverlag, Bibliographisches Institut, Mannheim, pp. 9–20.

Foulis, D. J., Greechie R. J., and Ruttimann G. T. (1992). Filters and supports in orthoalgebras, *International Journal of Theoretical Physics,* **21**, 789–807.

Giuntini, R. (1995) Quasilinear QMV algebras, *International Journal of Theoretical Physics,* **34**, 1397–1407.

Greechie R., Gudder, S. (n.d.) Effect algebra counterexamples, manuscript (1995).

Halmos, P. R. (1950). *Measure Theory,* Van Nostrand, Princeton, New Jersey.

Halmos, P. R. (1957). *Introduction to Hilbert Space, and the Theory of Spectral Multiplicity,* Chelsea, New York.

Halmos, P. R. (1962). *Algebraic Logic*, Chelsea, New York.

Kopka, F., and Chovanec, F. (1994). D-posets, *Mathematics Slovaca,* **44**, 21–34

Moisil, G. C. (1940). Recherches sur les logiques non-chrysippiennes, *Annals Scientifiques de l'Université de Jassy,* **26**, 431–466.

Moisil, G. C. (1941) Notes sur les logiques nonchrysippiennes, *Annals Scientifiques de l'Université de Jassy,* **27**, 86–98.

Pawlak, Z. (1982). Rough sets, *International Journal of Computer and Information Sciences,* **11**, 341–356.

Pawlak, Z, (1985). Rough sets and fuzzy sets, *Fuzzy Sets and Systems,* **17**, 99–102.